Some Results on Divisor Cordial Labeling of Graphs

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Abstract

In this paper, the divisor cordial labeling of \(S'(K_{2,m})\), \(S'(K_{1,n,n})\), double fan \(P_n+2K_1\), cone \(C_n+2K_1\) and \((P_n\cup P_m)+2K_1\) are presented.

Key words: Cordial labeling, Cordial Graphs, Divisor Cordial labeling, Divisor Cordial Graphs.

1. Introduction

By a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [4]. For standard terminology and notations related to number theory we refer to Burton [2] and graph labeling, we refer to Gallian [3]. In [1], Cahit introduce the concept of cordial labeling of graph. In [14], Varatharajan et al. introduce the concept of divisor cordial labeling of graph. The divisor cordial labeling of various types of graph are presented in [5-13,15]. The brief summaries of definition which are necessary for the present investigation are provided below.

2. Definitions

Definition :2.1

A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.

Definition :2.2

A mapping \( f : V(G) \rightarrow \{0,1\} \) is called binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \). If for an edge \( e = uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0,1\} \) is given by \( f^*(e) = |f(u) - f(v)| \). Then \( v_i^f(i) = \text{number of vertices of having label } i \text{ under } f \) and \( e_i^f(i) = \text{number of edges of having label } i \text{ under } f^* \).

Definition :2.3

A binary vertex labeling \( f \) of a graph \( G \) is called a cordial labeling if \( |v_i(0) - v_i(1)| \leq 1 \) and \( |e_i(0) - e_i(1)| \leq 1 \). A graph \( G \) is cordial if it admits cordial labeling.

Definition :2.4

Let a and b be two integers. If a divides b means that there is a positive integer k such that \( b = ka \). It is denoted by \( a \mid b \). If a does not divide b, then we denote \( a \notmid b \).

Definition :2.5

Let \( G = (V(G), E(G)) \) be a simple graph and \( f : V(G) \rightarrow \{1,2,\ldots,|V(G)|\} \) be a bijection. For each edge \( uv \), assign the label 1 if \( f(u) \mid f(v) \) or \( f(v) \mid f(u) \) and the label 0 otherwise. The function \( f \) is called a divisor cordial labeling if \( |e_i(0) - e_i(1)| \leq 1 \). A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition :2.6

For a graph \( G \), the splitting graph \( S'(G) \) of a graph \( G \) is obtained by adding a new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \).

Definition :2.7

The graph \( P_m (+) K_n \) is a graph with the vertex set \( V(G) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \) and the edge set \( E(G) = \{u_i u_{i+1}, u_i v_j, u_m v_j : 1 \leq i \leq m-1, 1 \leq j \leq n\} \).

Definition :2.8

The join \( G_1 + G_2 \) of \( G_1 \) and \( G_2 \) consists of \( G_1 \cup G_2 \) and all lines joining \( V_1 \) with \( V_2 \). The graph \( P_n + 2K_1 \) is called the double fan. The graph \( C_n + 2K_1 \) is called the double cone.
3. Main Results

Theorem 3.1

The graph \( S'(K_{2,m}) \) is divisor cordial graph.

Proof:

Let \( x_1, x_2, v_1, v_2, \ldots, v_m \) be the vertices of \( K_{2,m} \).

Then \( x_1, x_2, v_1, v_2, \ldots, v_m, x'_1, x'_2, v'_1, v'_2, \ldots, v'_m \) are the vertices of \( S'(K_{2,m}) \) and \( |V(G)| = 2m + 4 \) and \( |E(G)| = 6m \).

\( p_1 \) and \( p_2 \) are the largest and next largest prime numbers.

Define \( f : V(G) \rightarrow \{1, 2, 3, \ldots, 2m + 4\} \) by

\[
\begin{align*}
&f(x_1) = 1, \\
&f(x_2) = p_1, \\
&f(x'_1) = 2, \\
&f(x'_2) = p_2, \\
&f(v_i) = 2 + 2i, \quad 1 \leq i \leq m. \\
&f(v'_m) = 2m + 4,
\end{align*}
\]

Label the vertices \( v'_1, v'_2, \ldots, v'_m \) with odd numbers from \( 3, 5, \ldots, 2m + 3 \) other than \( p_1 \) and \( p_2 \).

Then \( e_f(0) = e_f(1) = 3m \) for any \( m \).

Therefore, \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( G \) is divisor cordial graph.

Example 3.1

The divisor cordial labeling of \( S'(K_{2,4}) \) is given in figure 3.1.

![Figure 3.1](image)

Theorem 3.2

The graph \( S'(K_{1,n,n}) \) is divisor cordial graph.

Proof.

Let \( u_1, u_2, \ldots, u_n \) be the vertices of path \( P_n \).

Then \( u_1, u_2, \ldots, u_n, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n} \) are the vertices of \( S'(K_{1,n,n}) \).

Then \( |V(G)| = 4n + 2 \) and \( |E(G)| = 6n \).

Define \( f : V(G) \rightarrow \{1, 2, 3, \ldots, 4n + 2\} \) by

\[
\begin{align*}
&f(v) = 1, \\
&f(u) = 2, \\
&f(u_i) = 4 + (i - 1), \quad 1 \leq i \leq n. \\
&f(v_{n+i}) = 5 + (i - 1), \quad 1 \leq i \leq n. \\
&f(v'_{n+i}) = 3 + (i - 1), \quad 1 \leq i \leq n.
\end{align*}
\]

Then \( e_f(0) = e_f(1) = 3n \) for any \( n \).

Therefore, \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( G \) is divisor cordial graph.

Example 3.2

The divisor cordial labeling of \( S'(K_{1,4,4}) \) is given in figure 3.2.

![Figure 3.2](image)

Theorem 3.3

The double fan \( P_n + 2K_1 \) is divisor cordial graph.

Proof.

Let \( u_1, u_2, \ldots, u_n \) be the vertices of path \( P_n \).

Then \( |V(G)| = n + 2 \) and \( |E(G)| = 3n - 1 \).

Define \( f : V(G) \rightarrow \{1, 2, 3, \ldots, n+2\} \) by

\[
\begin{align*}
&f(v) = 1, \\
&f(u) = 2, \\
&f(u_i) = 3 + (i - 1), \quad 1 \leq i \leq n.
\end{align*}
\]

Thus, \( n \) is odd, then \( e_f(0) = e_f(1) = \frac{3n - 1}{2} \) and \( e_f(1) = \frac{3n}{2} \).

Therefore, \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( G \) is divisor cordial graph.
Example 3.3

The divisor cordial labeling of $P_5 + 2K_1$ is given in figure 3.3.

![Figure 3.3](image)

Theorem 3.4

The double cone $C_n + 2K_1$ is divisor cordial graph.

Proof.

Let $u_1, u_2, \ldots, u_n$ be the vertices of path $C_n$.

Let $G$ be a graph $C_n + 2K_1$.

Let $V(G) = \{u_i, v, w: 1 \leq i \leq n\}$ and

$E(G) = \{u_iu_{i+1}, vu_j, wu_j, u_nu_1: 1 \leq i \leq n-1, 1 \leq j \leq n\}$.

Then $|V(G)| = n+2$ and $|E(G)| = 3n$.

Define $f: V(G) \rightarrow \{1, 2, \ldots, n+2\}$ by

$f(v) = 1$,

$f(w) = 2$,

$f(u_i) = p$, were $p$ is the largest prime and $p \leq n+2$.

Label the vertices $u_1, u_2, \ldots, u_{n-1}$ continuously with numbers from 3 to $n+2$ other than $p$.

Thus, $n$ is odd, then $e_f(0) = \frac{3n+1}{2}$ and $e_f(1) = \frac{3n-1}{2}$

and $n$ is even, then $e_f(0) = e_f(1) = \frac{3n}{2}$.

Therefore, $|e_f(0) - e_f(1)| \leq 1$.

Hence $G$ is divisor cordial graph.

Example 3.5

The divisor cordial labeling of $J_5$ is shown in figure 3.5.

![Figure 3.4](image)

Theorem 3.5

The Jewel graph $J_n$ is divisor cordial graph.

Proof.

Let $G$ be Jewel graph $J_n$.

Let $V(J_n) = \{u, x, v, y, u_i: 1 \leq i \leq n\}$ and

$E(J_n) = \{ux, vx, uy, xy, uu_i, vui: 1 \leq i \leq n\}$.

Then $|V(G)| = n+4$ and $|E(G)| = 2n+5$.

Define $f: V(J_n) \rightarrow \{1, 2, \ldots, n+4\}$ as follows

$f(x) = 2$,

$f(y) = 3$,

$f(u) = 1$,

and $f(v) = p$, where $p$ is the largest prime number and $p \leq n+4$ and label the vertices $u_1, u_2, \ldots, u_n$ with numbers from 4 to $n+4$ other than $p$.

Thus, $e_f(0) = n+3$ and $e_f(1) = n+2$.

Therefore, $|e_f(0) - e_f(1)| \leq 1$.

Hence $G$ is divisor cordial graph.

Example 3.6

The graph $P_m (+) \overline{K}_n$ is divisor cordial graph.

Proof.

Let $u_1, u_2, \ldots, u_m$ be the vertices of the path $P_m$.

Let $v_1, v_2, \ldots, v_n$ be the vertices of the path $\overline{K}_n$.

Let $G = P_m (+) \overline{K}_n$ be the graph with the vertex set

$V(G) = \{u_i, v_j: 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set

$E(G) = \{u_iu_{i+1}, u_1v_1, u_mv_j: 1 \leq i \leq m-1, 1 \leq j \leq n\}$.

Then $|V(G)| = m+n$ and $|E(G)| = m+2n-1$.

Define $f: V(G) \rightarrow \{1, 2, \ldots, m+n\}$ as follows

$p$ is the largest prime number and $p \leq m+n$.

Case (i) : $p \leq m-1$.
f(u_m) = p and label the vertices u_1, u_2, ..., u_{m-1} with following order other than p.

1, 2, 2^2, ..., 2^{k_1},
3, 3 \times 2 \times 2^2, ..., 3 \times 2^{k_2},
5, 5 \times 2 \times 2^2, ..., 5 \times 2^{k_2},
... ... ... ... ...
where (2t-1)2^{k_i} \leq m and t \geq 1, k_i \geq 0.

Observe that (2t-1)2^a divides (2t-1)2^b (a < b) and (2t-1)2^{k_i} does not divide (2t+1) and (2t+3).

In the above labeling, see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge.

Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

f(v_j) = m + j, 1 \leq j \leq n
Case (ii) : m \leq p \leq m+n.

f(u_m) = p and label the vertices u_1, u_2, ..., u_{m-1} with following order.

1, 2, 2^2, ..., 2^{k_1},
3, 3 \times 2 \times 2^2, ..., 3 \times 2^{k_2},
5, 5 \times 2 \times 2^2, ..., 5 \times 2^{k_2},
... ... ... ... ...
where (2t-1)2^{k_i} \leq m-1 and t \geq 1, k_i \geq 0.

Observe that (2t-1)2^a divides (2t-1)2^b (a < b) and (2t-1)2^{k_i} does not divide (2t+1) and (2t+3).

In the above labeling, see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge.

Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

Label the vertices v_1, v_2, ..., v_n with numbers from m+1 to m+n other than p.

In both cases, m is odd, then e_i(0) = e_i(1) = \frac{2n+m-1}{2} and m is even, then e_i(0) = \frac{2n+m-2}{2} and e_i(1) = \frac{2n+m}{2}.

Therefore, |e_i(0) - e_i(1)| \leq 1.

Hence G is divisor cordial graph.

Example 3.6

The divisor cordial labeling of P_7 (+) K_3 is shown in figure 3.6.

![Figure 3.6](image)

Theorem 3.7

The graph (K_n \cup P_m) + 2K_1 is divisor cordial graph.

Proof:

Let x, y, u_1, u_2, ..., u_k and v_1, v_2, ..., v_m be the vertices of K_n and P_m respectively.

Let G be the graph (K_n \cup P_m) + 2K_1 and let vertex set V(G) = \{x,y,u_i,v_j: 1 \leq i \leq n, 1 \leq j \leq m\} and the edge set E(G) = \{xu_i,yu_i, v_kv_{k+1}, xv_j, yv_j: 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq m-1\}.

Then |V(G)| = n+m+2 and |E(G)| = 2n+3m-1.

Define vertex labeling f: V(G) \rightarrow \{1, 2, ..., n+m+2\} as follows

f(x) = 1, f(y) = p, where p is the largest prime number.
Case (i) : p \leq m+1.

Label the vertices v_1, v_2, ..., v_m with following order other than 1 and p.

1, 2, 2^2, ..., 2^{k_1},
3, 3 \times 2 \times 2^2, ..., 3 \times 2^{k_2},
5, 5 \times 2 \times 2^2, ..., 5 \times 2^{k_2},
... ... ... ... ...
where (2t-1)2^{k_i} \leq m+2 and t \geq 1, k_i \geq 0.

Observe that (2t-1)2^a divides (2t-1)2^b (a < b) and (2t-1)2^{k_i} does not divide (2t+1) and (2t+3).

In the above labeling, see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge.

Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

Label the vertices v_1, v_2, ..., v_m with following order other than 1 and p.

1, 2, 2^2, ..., 2^{k_1},
3, 3 \times 2 \times 2^2, ..., 3 \times 2^{k_2},
5, 5 \times 2 \times 2^2, ..., 5 \times 2^{k_2},
... ... ... ... ...
where (2t-1)2^{k_i} \leq m+2 and t \geq 1, k_i \geq 0.

Observe that (2t-1)2^a divides (2t-1)2^b (a < b) and (2t-1)2^{k_i} does not divide (2t+1) and (2t+3).

In the above labeling, see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge.

Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

Therefore, |e_i(0) - e_i(1)| \leq 1.

Hence G is divisor cordial graph.
having labels even and odd numbers contribute 0 to each edge.

Label the vertices \( u_1, u_2, \ldots, u_n \) from \( m+3 \) to \( n+m+2 \).

Thus, \( n \) is odd, then \( e_f(0) = e_f(1) = \frac{2n+3m-1}{2} \) and \( n \) is even, then \( e_f(0) = \frac{2n+3m}{2} \) and \( e_f(1) = \frac{2n+3m-2}{2} \).

Therefore, \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( G \) is divisor cordial graph.

Example 3.7

The divisor cordial labeling of \( (\overline{K}_4 \cup P_4) \cup 2K_1 \) is given in figure 3.7.

![Diagram](image)

Theorem 3.8

The graph \( (P_n \cup P_m) + 2K_1 \) is divisor cordial graph, where \( n, m \geq 2 \).

Proof:

Let \( x, y, u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_m \) be the vertices of \( P_n \) and \( P_m \) respectively.

Let \( G \) be the graph \( (P_n \cup P_m) + 2K_1 \) and let vertex set \( V(G) = \{x, y, u_i, v_j \mid 1 \leq i \leq n, 1 \leq j \leq m\} \) and the edge set \( E(G) = \{xu_i, yu_i, u_iu_{i+1}, v_kv_{k+1}, xv_j, yv_j \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq r \leq n-1, 1 \leq k \leq m-1\} \).

Then \( |V(G)| = n+m+2 \) and \( |E(G)| = 3n+3m-2 \).

Define vertex labeling \( f : V(G) \to \{1, 2, \ldots, n+m+2\} \) as follows

\[
f(x) = 1,
\]
\[
f(y) = p, \text{ where } p \text{ is the largest prime number.}
\]

Label the vertices \( u_2, \ldots, u_n, v_1, v_2, \ldots, v_m \) in the following order other than \( p \) and \( n+m+1 \).

\[
1, 2, 2^2, \ldots, 2^{k_1},
\]
\[
3, 3 \times 2, 3 \times 2^2, \ldots, 3 \times 2^{k_2},
\]
\[
5, 5 \times 2, 5 \times 2^2, \ldots, 5 \times 2^{k_3},
\]
\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]
\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]

where \( (2t-1)2^{k_i} \leq m+1 \) and \( t \geq 1, k_i \geq 0 \).

Observe that \( (2t-1)2^a \) divides \( (2t-1)2^b \) \((a < b)\) and \( (2t-1)2^{k_i} \) does not divide \( (2t+1) \) and \( (2t+3) \).

In the above labeling, see that the consecutive adjacent vertices having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge.

Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even numbers contribute 0 to each edge.

Case (i) : \( n+m \) is odd and \( f(v_1) \) is even.

Then, \( e_f(0) = e_f(1) = \frac{2n+3m-1}{2} \).

Case (ii) : \( n+m \) is odd and \( f(v_1) \) is odd.

Then, \( e_f(1) = e_f(0)+1 = \frac{3n+3m-1}{2} \).

Case (iii) : \( n+m \) is even and \( f(v_1) \) is even.

Then, \( e_f(0) = e_f(1) = \frac{3n+3m-2}{2} \).

Case (iv) : \( n+m \) is even and \( f(v_1) \) is odd.

Subcase (a) : \( n+m = 6 \) and \( f(v_1) \) is odd.

Interchange the labels of \( u_1 \) and \( v_1 \). Then, \( e_f(0) = e_f(1) = 8 \).

Subcase (b) : \( n+m \neq 6 \) and \( f(v_1) \) is odd.

Interchange the labels of \( v_1 \) and \( v_m \).
Then, \(e_f(0) = e_f(1) = \frac{3n+3m-2}{2}\).

Therefore, \(|e_f(0) - e_f(1)| \leq 1\).

Hence \(G\) is divisor cordial graph.

Example 3.8

The divisor cordial labeling of \((P_4 \cup P_4) + 2K_1\) is given in figure 3.8.

4. Conclusions

In this paper, we prove the divisor cordial labeling of \(S'(K_{2,m})\), \(S'(K_{1,n,n})\), double fan \(P_n+2K_1\), cone \(C_n+2K_1\), Jewel graph \(J_n\), \(P_m (+) K_n\), \((K_n \cup P_m) + 2K_1\) and \((P_n \cup P_m)+2K_1\).

References