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A Common Fixed Point Theorem for Φ - Weak Contractive Maps

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Abstract: Common fixed point result is presented for Suzuki type φ - weak contractive maps in complete metric spaces, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$. Our results extend previous results of Zhang and Song (2009), as well as of Kikkawa and Suzuki (2008), Rhoades (2001), Nadler (1969) and Daffer and Kaneko (1995).

MSC: 54H25 and 47H10.

Keywords: Hausdorff metric; φ - weak contractive; lower semi-continuous; metric completeness; Banach contraction principle; Nadler's fixed point theorem; common fixed point.

1. Introduction

Throughout this paper we denote by N the set of all positive integers.

The Banach contraction theorem and its several extensions have been generalized using recently developed notion of weakly contractive maps. The following basic result is due to Rhoades [5].

Theorem 1.1. Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$.

Then T has a unique fixed point.

Very recently, the following theorem was given by Suzuki [7], which is a new type of generalization of the Banach contraction principle [1] (see also [3], [8]).

Theorem 1.2 Let (X, d) be a complete metric space and let $T : X \to X$. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$d(x, Tx) \leq (1+r)\, d(x, y) \text{ implies } d(Tx, Ty) \leq r\, d(x, y).$$

Then there exists $z \in X$ such that $z \in Tz$.

Definition 1.3 [6] Let (X, d) be a metric space. Two mappings $S, T : X \to X$ are called φ - weak contractive if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for every $x, y \in X$,

$$d(Sx, Ty) \leq d(x, y) - \varphi(d(x, y)).$$

The concept of $\varphi-$weak contractive mappings was defined by Daffer and Kaneko [2] in 1995.

The purpose of this paper is to combine the ideas of Theorem 1.1 and Theorem 1.2 and to obtain common fixed point theorems for a pair of maps in complete metric spaces.

2. Main Results

In this paper, we prove common fixed point theorem for $\varphi-$weak contractive mappings with constants in complete metric spaces as follows:

Theorem 2.1 Let (X, d) be a complete metric space and let $S, T : X \to X$. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\min\{d(x, Sx), d(y, Ty)\} \leq (1+r)\, d(x, y) \text{ implies}$$

$$d(Sx, Ty) \leq d(x, y) - \varphi\, d(x, y), \tag{2.1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$.

Then there exists $z \in X$ such that $z \in Sz \cap Tz$.

Proof. Take $x_0 \in X$. Putting $x_1 = Tx_0$ and $x_2 = Sx_1$, then let $x_3 = Tx_2$ and $x_4 = Sx_3$.

Inductively, Choose a sequence $\{ x_n \}$ in X so that

$$x_{2n+1} = Tx_{2n} \text{ and } x_{2n+2} = Sx_{2n+1} \text{ for all } n \geq 0.$$

As $d(x_{n-1}, x_n) \leq (1+r)\, d(x_{n-1}, x_n)$ (2.2)

Now if n is odd

Suppose if $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ (2.3)

Then by (2.2) and (2.3)

$$\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leq (1+r)\, d(x_{n-1}, x_n)$$

And this implies (2.1), that is, we have

$$d(Sx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}) - \varphi\ d(x_n, x_{n+1}).$$

Suppose if $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, (2.4)

and by (2.2), $d(x_{n-1}, x_n) \leq (1+r)\, d(x_{n-1}, x_n)$.

So $d(x_n, x_{n+1}) \leq (1+r)\, d(x_{n-1}, x_n)$. (2.5)

Then by (2.4) and (2.5)

$$\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leq (1+r)\, d(x_{n-1}, x_n)$$

And this implies (2.1), that is, we have

$$d(Sx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) - \varphi\ d(x_n, x_{n-1}).$$

It follows from property of the function φ that if n is an odd,

$$d(x_{n+1}, x_n) = d(Sx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1}))$$

$$\leq d(x_n, x_{n-1}).$$

i.e. $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$.

Similarly if n is even, we obtain $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$.

Therefore, for all $n \geq 0$, $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$ and so $\{d(x_{n+1}, x_n)\}$ is monotonic nonincreasing and bounded below, so their exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r = \lim_{n\to\infty} d(x_n, x_{n-1}) \tag{2.6}$$

Then (by lower semi-continuity of φ)

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi\ (d(x_n, x_{n-1})).$$

We claim that $r = 0$. Infact taking upper limits as $n \to \infty$ on either side of the following inequality:

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1})).$$

and using (2.6),We have

$r \leq r - \varphi(r)$.

i.e. $\varphi(r) \leq 0$. Then $\varphi(r) = 0$ by the property of function φ, and furthermore by property of function φ

$\varphi(r) = 0$ implies $r = 0$.

So $\lim_{n\to\infty} d(x_{n+1}, x_n) = r = 0.$ (2.7)

Next we claim that $\{x_n\}$ is Cauchy.

Let $C_n = \sup\{d(x_j, x_k) : j, k \geq n\}$.

Then $\{C_n\}$ is decreasing.

If $\lim_{n\to\infty} C_n = 0$, then we are done.

Assume that $\lim_{n\to\infty} C_n = C > 0$.

Choose $\varepsilon < \frac{C}{8}$ small enough and select N such that for all $n \geq N$,

$d(x_{n+1}, x_n) < \varepsilon$ and $C_n < C + \varepsilon$.

By the definition of C_{N+1}, there exists $m, n \geq N + 1$ such that

$$d(x_m, x_n) > C_n - \varepsilon \geq C - \varepsilon.$$

Replacing x_m by x_{m+1} if necessary, we have

$$d(x_n, x_{m+1}) > C - \varepsilon \tag{2.8}$$

i.e. $d(x_n, x_{m+1}) - d(x_{m+1}, x_m) > C - \varepsilon - d(x_{m+1}, x_m)$

i.e. $d(x_n, x_m) \geq d(x_n, x_{m+1}) - d(x_{m+1}, x_m) > C - \varepsilon - d(x_{m+1}, x_m)$

i.e. $d(x_m, x_n) > C - \varepsilon - d(x_{m+1}, x_m)$

i.e. $d(x_m, x_n) > C - \varepsilon - \varepsilon$

i.e. $d(x_m, x_n) > C - 2\varepsilon.$ (2.9)

We may assume that m is even, n is odd

Then $d(x_{m-1}, x_{n-1}) > C - 4\varepsilon$.

And since $d(x_{m-1}, x_m) \leq d(x_{m-1}, x_{n-1})$ and $d(x_{n-1}, x_n) \leq d(x_{m-1}, x_{n-1})$.

So $\min\{d(x_{m-1}, x_m), d(x_{n-1}, x_n)\} \leq d(x_{m-1}, x_{n-1})$.

i.e. $\min\{d(x_{m-1}, x_m), d(x_{n-1}, x_n)\} \leq (1+r)\, d(x_{m-1}, x_{n-1})$.

i.e. $\min\{d(x_{m-1}, Sx_{m-1}), d(x_{n-1}, Tx_{n-1})\} \leq (1+r)\, d(x_{m-1}, x_{n-1})$.

So from given assumption

$$d(Sx_{m-1}, Tx_{n-1}) \le d(x_{m-1}, x_{n-1}) - \varphi(d(x_{m-1}, x_{n-1})).$$

i.e. $d(x_m, x_n) = d(Sx_{m-1}, Tx_{n-1}) \le d(x_{m-1}, x_{n-1}) - \varphi(d(x_{m-1}, x_{n-1})).$

i.e. $d(x_m, x_n) \le d(x_{m-1}, x_{n-1}) - \varphi(d(x_{m-1}, x_{n-1})).$

We have proved that $C_{N+1} < C_N - \varphi\left(\frac{C}{2}\right)$. (if ε is small enough)

This is impossible. Thus we must have $C = 0$.

That is, the sequence $\{x_n\}$ is Cauchy sequence. Since X is complete, so the sequence $\{x_n\}$ is convergent.

i.e. there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Moreover $x_{2n} \to z$ and $x_{2n+1} \to z \;\; n \to \infty$.

Now we prove that z is fixed point of T.

Since $x_n \to z$, there exists $n_0 \in N$ such that

$$d(z, x_n) \le \frac{1}{3} d(z, y) \text{ for all } n \in N \text{ with } n \ge n_0.$$

Then we have

$$\begin{aligned}(1+r)^{-1} d(x_{2n-1}, Sx_{2n-1}) \le d(x_{2n-1}, Sx_{2n-1}) &\le d(x_{2n-1}, x_{2n}) \\ &\le d(x_{2n-1}, z) + d(z, x_{2n}) \\ &\le \frac{2}{3} d(y, z) = d(y, z) - \frac{1}{3} d(y, z) \\ &\le d(y, z) - d(x_{2n-1}, z) \\ &\le d(x_{2n-1}, y)\end{aligned}$$

$$d(x_{2n-1}, Sx_{2n-1}) \le (1+r)\, d(x_{2n-1}, y). \tag{2.10}$$

Now suppose if $d(y, Ty) \le d(x_{2n-1}, Sx_{2n-1})$.

Then $d(y, Ty) \le d(x_{2n-1}, Sx_{2n-1}) \le (1+r)\, d(x_{2n-1}, y)$.

And this implies (2.1), that is, we have

$$d(Sx_{2n-1}, Ty) \le d(x_{2n-1}, y) - \varphi(d(x_{2n-1}, y)).$$

Letting $n \to \infty$, we have

$$d(z, Ty) \leq d(z, y). \tag{2.11}$$

Suppose if $d(x_{2n-1}, Sx_{2n-1}) \leq d(y, Ty)$,

and this implies (2.1), that is, we have

$$d(Sx_{2n-1}, Ty) \leq d(x_{2n-1}, y) - \varphi(d(x_{2n-1}, y)).$$

Letting $n \to \infty$, we have

$$d(z, Ty) \leq d(z, y). \tag{2.12}$$

Thus from (2.11) and (2.12), we conclude

$$d(z, Ty) \leq d(z, y) \text{ for all } y \in X - \{z\}. \tag{2.13}$$

We next prove that

$$d(Sz, Ty) \leq d(z, y). \tag{2.14}$$

We assume that $y \neq z$.

Then for every $n \in N$, there exists $z_n \in Ty$ such that

$$d(z, z_n) \leq d(z, Ty) + \frac{1}{n} d(z, y).$$

Now $d(y, Ty) \leq d(y, z_n) \leq d(y, z) + d(z, z_n)$

$$\leq d(y, z) + d(z, Ty) + \frac{1}{n} d(z, y)$$

$$\leq d(y, z) + d(y, z) + \frac{1}{n} d(z, y)$$

i.e. $d(y, Ty) \leq \left(2 + \frac{1}{n}\right) d(y, z)$

i.e. $\frac{1}{2} d(y, Ty) \leq d(y, z)$.

Suppose if $d(y, Ty) \leq d(z, Sz)$,

and this implies (2.1), that is, we have

$$d(Sz, Ty) \leq d(z, y) - \varphi(d(z, y)). \tag{2.15}$$

And if $d(z, Sz) \leq d(y, Ty)$

i.e. $\frac{1}{2} d(z, Sz) \leq \frac{1}{2} d(y, Ty)$

i.e. $\frac{1}{2} d(z, Sz) \leq d(z, y)$.

And this implies (2.1), that is, we have

$$d(Sz, Ty) \leq d(z, y) - \varphi(d(z, y)). \tag{2.16}$$

Thus from (2.15) and (2.16), we have

$$d(Sz, Ty) \leq d(z, y) - \varphi(d(z, y)).$$

Finally, Since

$$\begin{aligned} d(z, Sz) &= \lim_{n\to\infty} d(x_{2n+1}, Sz) \\ &= \lim_{n\to\infty} d(Tx_{2n}, Sz) \\ &\leq \lim_{n\to\infty} [d(z, x_{2n}) - \varphi(d(z, x_{2n}))] \\ &= d(z, z) - \varphi(d(z, z)) = 0, \end{aligned}$$

i.e. $d(z, Sz) = 0$.

implies $z = Sz$.

Similarly we can prove that $z = Tz$.

Thus $z = Sz = Tz$.

Corollary 2.1. Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,

$$d(x, Tx) \leq (1+r)\, d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

where φ is defined as in Theorem 2.1. Then T has a unique fixed point.

Proof. It comes from Theorem 2.1 when $S = T$.

The following example shows the generality of our results.

Example 2.1 Define a complete metric space X by

$X = \{(0, 0), (0, 4), (4, 0), (4, 5), (5, 4)\}$

and its metric d by

$$d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Let S, T and f be such that

$$S(x_1, x_2) = \begin{cases} (0, x_1) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2, \end{cases} \qquad T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2 \end{cases}$$

Then S and T satisfy the assumption in Theorem 2.1 (and also Corollary 2.1), but do not satisfy the assumption in Theorem 1.1 at $x = (4, 5)$, $y = (5, 4)$.

Proof. We first note that $d(Sx, Ty) \leq d(x, y) - \varphi(d(x, y))$

if $(x, y) \neq ((5, 4), (4, 5))$. Since at $(x, y) = ((5, 4), (4, 5))$

$\min\{d(x, Sx), d(y, Ty)\} = \min\{d((5, 4), S(5, 4)), d((4, 5), T(4, 5))\}$

$$= \min\{5, 5\} = 5 > (1+ r)\, 2 = (1+ r)\, d(x, y).$$

Thus S and T satisfy the assumption in Theorem 2.1 (and also Corollary 2.1), but do not satisfy the assumption in Theorem 1.1.

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