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PAIRABLE GRAPHS

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Abstract

Let G(V,E) be a connected graph. For a vertex v in V, the set of all neighbours of v is called an open neighbourhood of v and is denoted by N(v). The closed neighbourhood of v is defined by $N[v] = N(v) \cup \{v\}$. A connected graph G is said to be neighbourhood highly irregular (or simply NHI) if for any vertex $v \in V$, any two distinct vertices in N(v) have distinct closed neighbourhood sets. In this paper, we prove some results on neighbourhood highly irregular graphs. We also introduce a new concept called pairable graphs. A pairable graph is defined as a graph in which for any vertex $u \in V$, there exists a vertex v distinct from u in V such that N[u] = N[v]. We study the properties of pairable graphs and obtain some results on them.

Keywords: Pairable vertices, Extreme vertices, NHI graphs, Pairable graphs, Splitting graphs, Super splitting graphs.

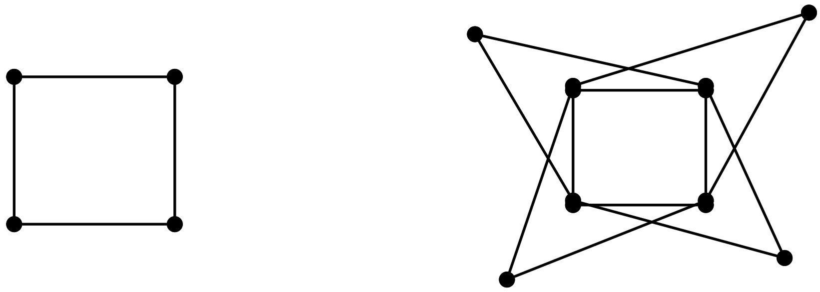
AMS Subject Classification Code(2000): 05C (Primary)

1 Introduction

Throughout this paper, we consider only finite, simple, undirected and connected graphs. For notations and terminology we follow [5]. Let G(V,E) be a connected graph of order n. For any vertex $v \in V$, the *open neighbourhood* N(v) of v is the set of all vertices adjacent to v. That is, $N(v) = \{u \in V / uv \in E\}$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. Clearly, if N[u] = N[v], then u and v are adjacent and have the same degree. A *full vertex* of G is a vertex in G which is adjacent to all other vertices of G. A *1 – factor* of G is a 1 – regular spanning subgraph of G and it is denoted by F. Let χ denote the chromatic number of a graph and let β denote its independence number. A vertex v in G is called *isomorphic image* of u if there is an automorphism f from G to G such that f(u) = v.

A vertex of degree one is said to be a *pendant vertex*. A non pendant vertex v is said to be an *extreme vertex* if < N(v) > is complete, where < N(v) > is the subgraph of G induced by N(v). Note that < N(v) > is complete if and only if < N[v] > is complete.

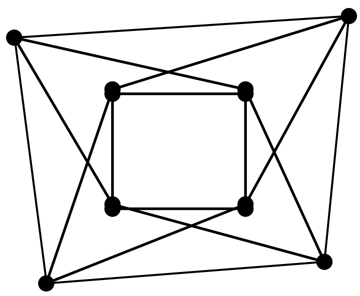
The concept of splitting graph was introduced by Sampath Kumar and Walikar[6]. The *splitting graph* S(G) of a graph G is the graph obtained from G, by taking a new vertex v′ for every vertex $v \in V$ and joining v′ to all vertices of G adjacent to v. A graph G is said to be a *splitting graph* if it is a splitting graph of some graph H. For example, a graph G and its splitting graph S(G) are shown in Figure 1.



G **S(G)**

Figure 1

For a graph G(V,E) with vertex set V(G) = {v_1, v_2, ….. , v_n}, the *super splitting graph SS(G)* of G is defined as the graph with vertex set V(SS(G)) = { u_1, u_2, ….. , u_n, w_1, w_2, ….. , w_n} and edge set E(SS(G)) = {u_iu_j, w_iw_j, u_iw_j / $v_iv_j \in$ E(G), $1 \le i, j \le n$}. For example, the super splitting graph of C_4 is shown in Figure 2.



SS(G)

Figure 2

A connected graph G is said to be *highly irregular* [2], if each of its vertices is adjacent only to vertices with distinct degrees. For example, the graphs shown in Figure 3 are highly irregular.

Figure 3

For further details on highly irregular graphs, one can refer [1], [2], [3] and [7].

The concept of neighbourhood highly irregular graphs has been introduced by V.Swaminathan and A.Subramanian [8]. A connected graph is said to be *neighbourhood highly irregular (NHI)* if for any vertex $v \in$ V, for any two distinct neighbours u and w of v, N[u] ≠ N[w], that is, if for any vertex v, any two distinct neighbours of v have distinct closed neighbourhood sets.

Any highly irregular graph is NHI, but there are NHI graphs which are not highly irregular. For example, any cycle of order at least four is NHI but not highly irregular.

A necessary and sufficient condition for a graph to be NHI is obtained by Selvam Avadayappan and P. Santhi in [4]. The various results discussed in [4] and [8] are given below:

Result 1[4] A connected graph G with $n \geq 3$ is NHI if and only if $N[u] \neq N[v]$, for any pair of adjacent vertices u and v in G with $d(u) = d(v)$.

Result 2[8] Every graph of order $n \geq 2$ is an induced sub graph of an NHI graph of order $2n – k$, where k is the number of pendant vertices of G.

Result 3[8] For $n \geq 3$, the smallest order of an NHI graph with clique number n is $2n – 1$.

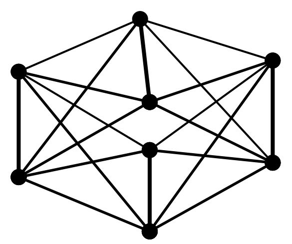
The bound mentioned in Result 3 is not sharp. Selvam Avadayappan and P. Santhi [4] obtained a sharp lower bound for the order of any NHI graph with clique number n, which is stated as follows:

Result 4[4] For any $n \geq 1$, the smallest order of an NHI graph with clique number n is $n + m$, where m is the least possible integer such that $n \leq 2^m$.

For any two distinct vertices u and v in G, u is said to be *pairable* with v if $N[u] = N[v]$ in G. A vertex in G is called a *pairable vertex* if it is pairable with a vertex in G. An equivalent definition for an NHI graph can be given as follows:

Any connected graph with at least 3 vertices is NHI if and only if it does not contain a pairable vertex.

A connected graph G of order at least 2 is said to be a *pairable graph* if every vertex of G is pairable. For example, K_n is a pairable graph of order n, for any $n \geq 2$. Clearly pairable graphs are not NHI. A 1 – factor F in a pairable graph is said to be a *pairing 1 – factor* if $E(F) = \{uv \in E(G) / u \text{ is pairable with } v \text{ in } G\}$. For example a pairable graph G and a pairing 1 – factor (shown in bold lines) are given in Figure 4.



G

Figure 4

In this paper, we construct an NHI graph of minimum order which contains any given graph as an induced subgraph. Also we prove that there is no pairable splitting graph. And we show that every graph is an induced subgraph of a pairable graph.

2 Main Results

Let $\mathscr{P}(G)$ be the set of all pairable vertices of a graph G. Then clearly $\mathscr{P}(G) \geq 2$, for any non NHI graph G. Define a relation ρ on $\mathscr{P}(G)$ by, u ρ v if and only if N[u] = N[v]. Then clearly ρ is an equivalence relation on $\mathscr{P}(G)$. Therefore ρ partitions $\mathscr{P}(G)$ into equivalence classes. Let them be $\mathscr{P}_1(G), \mathscr{P}_2(G), \ldots, \mathscr{P}_m(G)$. Now for any i, $1 \leq i \leq m$, if u and v are in $\mathscr{P}_i(G)$, then N[u] = N[v] and hence u and v are adjacent in G. This forces that $\mathscr{P}_i(G)$ induces a complete graph for every i, $1 \leq i \leq m$.

Let us now construct an NHI graph of minimum order which contains any given graph as an induced subgraph.

Theorem 1 Any graph G of order n is an induced subgraph of an NHI graph of order at least n + p, where p is the least possible integer such that $\max_i \{|\mathscr{P}_i(G)|\} \leq 2^p$.

Proof If the graph G itself is an NHI graph, then there is nothing to prove. If not, then $|\mathscr{P}(G)| \geq 2$. Then we can define a relation ρ on $\mathscr{P}(G)$ as stated above. Let $\mathscr{P}_i(G)$ be the equivalence classes of ρ, where $1 \leq i \leq m$, for some m. Then each $\langle \mathscr{P}_i(G) \rangle$ is a complete graph**.** We aim to add new vertices in G, such that the resultant graph contains no two vertices u and v such that N[u] = N[v] for every u, v $\in \mathscr{P}_i(G)$, and for every $1 \leq i \leq m$. Let $q_i = |\mathscr{P}_i(G)|$ and q = max $\{q_1, q_2, \ldots, q_m\}$. Let u_{ij}, $1 \leq i \leq m$ and $1 \leq j \leq q_i$ denote the vertices in $\mathscr{P}_i(G)$.

Now we construct a graph H with the vertex set $V(H) = V(G) \cup \{v_1, v_2, \ldots, v_p\}$ where p is the least possible integer such that $q \leq 2^p$ and the edge set $E(H) = E(G) \cup \{u_{ij}v_{j-1} / 1 \leq i \leq m;\ 1 < j \leq q_i\} \cup \{u_{ij}v_{r_1}, u_{ij}v_{r_2}, \ldots, u_{ij}v_{r_k} / 1 \leq i \leq m;\ 1 \leq k \leq q_i;\ q_iC_{k-1} < j \leq q_iC_k;\ r_a \neq r_b$ for $a \neq b$, $1 \leq a, b \leq k$ such that the unordered tuple $(r_1, r_2, \ldots, r_k)$ is distinct.$\}$.

Next we claim that the constructed graph H is an NHI graph. That is, we have to prove that $\mathscr{P}(H) = \phi$. Since no new edges are added with the non – pairable vertices of G, it is enough to check the pairability of the vertices in $\mathscr{P}(G) \cup \{v_1, v_2, \ldots, v_q\}$. By our construction, the neighbour set of each vertex in any equivalence class of ρ differs from all other vertices in the same class by at least one. Hence they are no longer pairable with each other. Also, $\{v_1, v_2, \ldots, v_q\}$ is independent and thereby those vertices are not pairable with each other. It remains to check the pairability of v_i with any of the vertex in $\mathscr{P}(G)$. This pairability is also not possible because no v_j has all vertices in $\mathscr{P}_i(G)$ as neighbours. Thus the resultant graph contains no pairable vertices.

Finally we claim that H is an NHI graph of minimum order which contains G as an induced subgraph. Suppose not, let H* be one such NHI graph of minimum order that contains G as an induced subgraph. Then |H*| = n + w, where w is a positive integer such that w < p. Now in H*, let us examine the nature of the induced subgraph (say H') of $\mathscr{P}_j(G)$ with q

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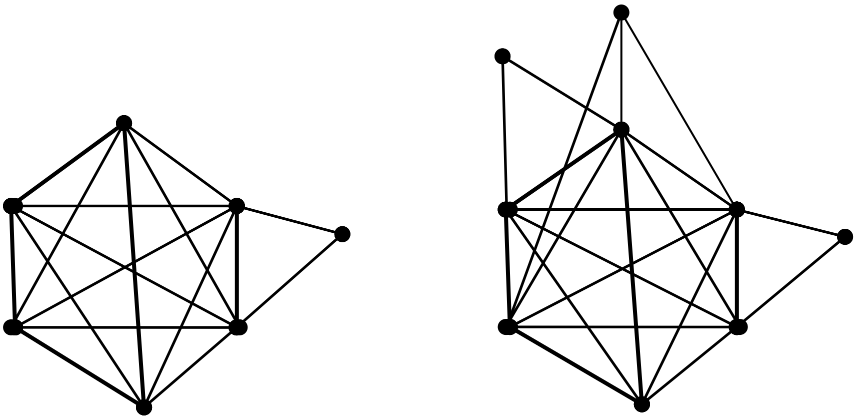
vertices along with the newly added vertices $v_1, v_2, ..., v_w$. The q vertices in $\mathscr{P}_j(G)$ are no more pairable vertices in H*. Therefore every two of these vertices has at least one different neighbour in H*. We know that $wC_0 + wC_1 + wC_2 + ... + wC_w = 2^w$. But $q > 2^w$. It is impossible for q vertices to have q different combinations from these w vertices. Therefore H* is not NHI which is a contradiction. Hence H is the required NHI of minimum order.

For example a graph G and the corresponding NHI graph H constructed are shown in Figure 5. ∎



G **H**

Figure 5

The following theorem discusses the nature of extreme vertices in an NHI graph.

Theorem 2 In any NHI graph G of order at least three, the set S of all extreme vertices and pendant vertices of G is independent.

Proof Let G be any NHI graph of order $n \geq 3$. If $|S| \leq 1$, then there is nothing to prove. Assume that S contains at least 2 vertices. Two pendant vertices are adjacent in G if and only if $G \cong K_2$, which is not the case. Also a pendant vertex cannot be a neighbour of an extreme vertex. Therefore it is enough if we prove that no two extreme vertices are adjacent in G.

If possible, let u and v be two extreme vertices which are adjacent in G. Then $< N(v) >$ and $< N(u) >$ are complete graphs. Since $u \in N(v)$, we have $N[u] = N[v]$. Therefore u and v are pairable vertices of G and so G is not NHI, a contradiction. Hence S is independent. ∎

The converse of the above theorem is not true. For example, consider the graph shown in Figure 6. Clearly v_3 and v_4 are pairable vertices and so G is not NHI. But $S = \{v_1, v_2, v_6, v_7, v_8\}$ is independent in G.

Corollary 3 If K_m ($m \geq 2$) is an induced subgraph of an NHI graph G of order $n \geq 3$, then K_m has at most one vertex of degree $m - 1$ in G.

Proof Suppose there are two vertices in K_m of degree $m - 1$ in G. Then they are adjacent extreme vertices of G, which is a contradiction to the above theorem. ■

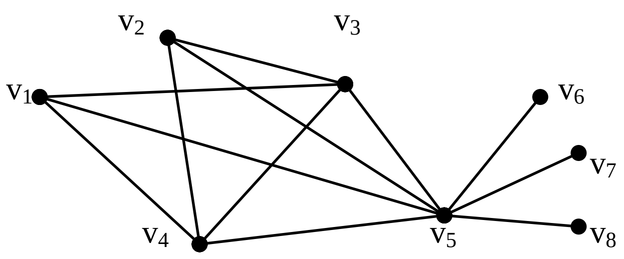


Figure 6

The converse of the above statement is also not true. That is for any induced subgraph K_m of G, if at most one of its vertices is of degree $m - 1$, then G need not be NHI. For example the graph G shown in Figure 6 is not NHI but it has no vertices of degree 2 in any K_3.

Let m denote the number of vertices in S. Then it can be easily verified that,

Corollary 4 For any NHI graph G, $\beta \geq m$. ■

Corollary 5 If G is an NHI graph, then $\chi(G) \leq n - m + 1$.

Proof By Corollary 4, the maximal independent set contains at least m vertices which can be assigned a same colour. If we assign distinct colours for the remaining $n - m$ vertices, then we have a proper $n - m + 1$ coloring of G. Thus $\chi(G) \leq n - m + 1$. ■

The inequality stated in Corollary 5 is strict, which has been proved in the following theorem with a simple construction.

Theorem 6 For any three positive integers p, q, r such that $r - p - 2 > q > (r - p)/2$, there exists an NHI graph G of order r with p extreme vertices and q pendant vertices such that $\chi(G) = r - (p+q) + 1$.

Proof Consider the graph $K_{r-(p+q)} \vee K_p^c$ with vertex set $\{v_1, v_2, \ldots, v_{r-(p+q)}; u_1, u_2, \ldots, u_p\}$. With this graph add pendant vertices $w_1, w_2, \ldots, w_q$ such that w_i is adjacent to $v_{i \bmod (r - (p+q))}$. Let the resultant graph be denoted by G. In G, no two adjacent vertices have common neighbours. Hence G is an NHI graph. And it can be easily verified that $\chi(G) = r - (p+q) + 1$.

For example, when p = 3, q = 4 and r = 12, the graph G constructed above is shown in Figure 7. ■

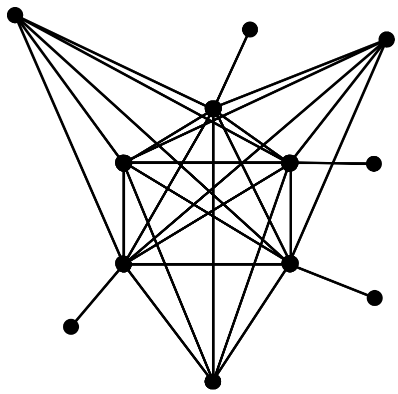


Figure 7

Theorem 7 A graph G of order n is NHI if and only if the matrix B = A + I has no two identical rows (columns), where A denotes the adjacency matrix of G and I is the identity matrix of order n.

Proof Let G be any NHI graph of order n. Then G has no pairable vertices. That is, we cannot find two vertices u and v in G such that N[u] = N[v]. Suppose the matrix B = A + I has two identical rows (columns). We know that A has two identical rows (columns) only when the corresponding vertices are not adjacent and they have same neighbours. The matrix B is obtained from the adjacency matrix A of G by replacing all diagonal entries with one. Therefore if B has two identical rows (columns), then the corresponding vertices are adjacent with same neighbours. That is, G contains two vertices u and v such that N[u] = N[v], which is a contradiction.

The converse is obvious. ∎

Theorem 8 Any super splitting graph contains no pairable vertices.

Proof Let G be a super splitting graph. We claim that G contains no pairable vertices. On contrary, assume that G contains a pairable vertex u. Since G is a splitting graph, corresponding to u, there is a vertex v in G such that N(u) = N(v) but u and v are not adjacent. Also since u is a pairable vertex, there is a vertex w distinct from u such that N[u] = N[w]. This means that w $\in$ N(v). Hence v and w are pairable. But u and w are pairable. Thus u and v are pairable and so they are adjacent, a contradiction. Therefore G contains no pairable vertices. Hence the proof. ∎

Corollary 9 Any super splitting graph is an NHI graph. ■

Theorem 10 Any graph G is an induced subgraph of a pairable graph.

Proof Let G be any graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider the super splitting graph SS(G) of G. Let $\{u_1, u_2, \ldots, u_n\}$ be the newly added vertices in SS(G) such that u_i corresponds to v_i. Then it is clear that $N(u_i) = N(v_i)$ for each i, $1 \leq i \leq n$. Now construct a graph H with the vertex set V(H) = V(SS(G)) and edge set $E(H) = E(SS(G)) \cup \{u_iv_i / 1 \leq i \leq n\}$. It is obvious that G is an induced subgraph of H. Also from our construction of H, it is clear that $N[u_i] = N[v_i]$ for each i, $1 \leq i \leq n$. Hence H is a pairable graph. ■

It is clear from the above theorem that for any graph G, the union of a 1 – factor with its super splitting graph results in a pairable graph. On the other hand, the removal of a 1 – factor from a pairable graph of even order need not result in a super splitting graph. The following theorem gives a necessary and sufficient condition for the existence of such graphs.

Theorem 11 Let G be a pairable graph of order 2n, where $n \geq 1$. Then G – F is a super splitting graph if and only if F is a pairing 1 – factor.

Proof Consider a pairable graph G of order 2n, $n \geq 1$. Then by Theorem 8, G cannot be a super splitting graph. Let F be a 1 – factor of G such that G – F is a super splitting graph. We claim that F is a pairing 1 – factor. If possible, let F contain an edge uv whose end vertices are not pairable with each other in G. Since G – F is a super splitting graph, corresponding to u there exists a vertex w in G such that N(u) = N(w) in G – F. Clearly w is distinct from v. Otherwise u and v are pairable with each other in G. Therefore u and w are not adjacent in G also.

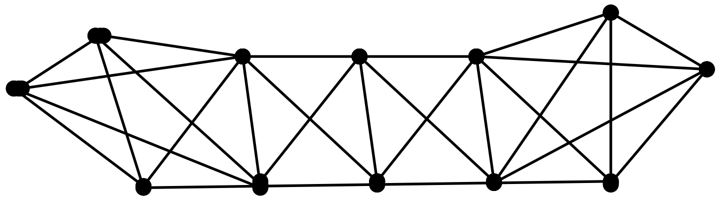
Now since G is a pairable graph, we conclude that at least one of the neighbours, say v_1, of u other than v is pairable with u. Hence $N[v_1] = N[u]$ in G. Since all neighbours of u other than v, are neighbours of w in G, we have $v_1 \in N(w)$. Therefore u and w are adjacent, which is a contradiction. Hence F cannot contain such an edge uv and thus F is a pairing 1 – factor.

Conversely suppose that G is a pairable graph in which a pairing 1 - factor F is removed. Then $N(u_i) = N(u_j)$ in G – F, for every $u_iu_j \in E(F)$ where $1 \leq i,j \leq 2n$, $i \neq j$. Hence G – F is a super splitting graph. ■

Theorem 12 For any $n \geq 3$, there exists a pairable graph G of order 2n which contains a 1 – factor whose removal does not result in a super splitting graph.

Proof Let $n \geq 3$ be any positive integer. Consider the graph G_{2n} with the vertex set $V(G_{2n}) = \{u_1, u_2, \ldots, u_{n-1}, v_0, v_1, \ldots, v_n\}$ and the edge set $E(G_{2n}) = \{u_iu_{i+1} / 1 \leq i \leq n-2\} \cup \{v_iu_j / 2 \leq i \leq n-2, i-1 \leq j \leq i+1\} \cup \{v_0v_1, v_1u_1, v_1u_2, v_{n-1}v_n, v_0u_1, v_0u_2, v_{n-1}u_{n-1}, v_{n-1}u_{n-2}, v_nu_{n-1}, v_nu_{n-2}\}$. In this graph G_{2n}, u_i is pairable with v_i for every i, $1 \leq i \leq n-1$. Also v_0 and v_n are pairable with v_1 and v_{n-1} respectively. Hence G_{2n} is a pairable graph. From our construction, G_{2n} contains no pairing 1 – factor. Hence the theorem.

For example, the graph G_{12} shown in Figure 8 is a pairable graph which contains no pairing 1 – factor F such that G – F is not a splitting graph. ■



G_{12}

Figure 8

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