IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 1 Issue 5, July 2014.
www.ijiset.com
ISSN 2348 – 7968

Record Values on The Size-Biased Student's t Distribution

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Abstract

Record values have useful applications in many real life data including weather, business, economics and sports. In this paper we discuss the upper record values arising from the size-biased student's t distribution. The moments, mean and variance of the upper record values from the size-biased student's t distribution have been derived. Moreover the cumulative distribution function, survival function, hazard function and entropy of the upper record values from the size-biased student's t distribution have been developed along with their graphs. We establish confidence intervals for estimating the parameter of upper record values from size-biased student's distribution.

Keywords: *Weighted Distributions, cdf, pdf, Moments, Entropy, Hazard Function.*

1. Introduction

Size biased distributions are special case of the weighted distributions. Such distributions rise certainly in practice when observations from a sample are recorded with unequal probability. The weighted distributions arise when the observations generated from a stochastic process, are recorded according to some weight function. When the weight function depends on the lengths of the items of concern then the subsequent distribution is called length biased. Patil and Rao (1978) examined some general models of weighted distributions when weight functions not necessarily restricted by unity. Several important distributions and their size-biased forms were discussed. Few theorems were given on the inequalities among the mean values of two weighted distributions. The results were applied to the data relating to human populations and wildlife management. Mir and Ahmad (2009) introduced some size-biased probability distributions and their generalizations. These distributions provide a solution to the problems where the observations fall in the non-experimental, non- replicated, and nonrandom categories. This paper also explored some of the possible uses of size-biased distribution theory to some real life data.

Record values are of great interest and importance in several real-life data involving weather, economic, athletic and sports. Record values appear in many statistical applications. Chandler (1952) introduced the record value theory. Ahsanullah (1991) derived some properties of the record values from the exponential distributions. Ahsanullah, et. al. (1992) established some recurrence relations between the moments of the record values from the Gumbel distribution. All the single and product moments of all record values can be obtained by using these relations. Ahsanullah and Balakrishnan (1994) derived some recurrence relations between the moments of record values from Lomax distribution. Raqab and Ahsanullah (2003) investigated distributional properties using relations between moment generating functions of the record values from the generalized extreme value distribution. Sultan (2007) established new recurrence relations between the single moments of record values from the modified Weibull distribution. They used the recurrence relations to calculate the mean, variance and product moments. Shakil and Ahsanullah (2011) considered the record values from the ratio of Rayleigh random variables. They derived the distributional properties as moments, cdf, survival and hazard functions, and entropy with graphs. . Shakila and Ahmad (2014) developed record values from the size-biased Pareto distributions and derived its various properties including a characterization. Shakila, Ahmed and Ahmad (2014) established some distributional properties of record values from the two-sided power distribution.

Suppose $X_1, X_2, \cdots$ are identically and independently distributed random variables with cumulative distribution function F(x). Let $Y_n = \max(\min)\{X_1, \cdots X_n\}, n \geq 1.$ We say that X_j is an upper (lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}, j > 1.$ By definition, X_1 is an upper record value as well as lower record value. We consider that the upper record values as record values unless it is otherwise mentioned.

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2. Size-biased student's t distribution

Weighted distributions used when the recorded observations are not generated randomly. Weighted distributions have a number of applications in forestry, in observational studies of human life, environment, insect, plant etc. Length-biased distributions have been used as weighted distributions in reliability perspective.

Dara and Ahmad (2012) Described the Size-Biased Student's t distribution, when $w(x) = x$. The probability density function (pdf) of the size biased Student's t Distribution

$$f(x) = \frac{(\nu-1)}{\nu} x \left(\frac{x^2}{\nu}+1\right)^{-\left(\frac{1+\nu}{2}\right)}, \quad x>0; \nu>0, \qquad (1)$$

with cumulative distribution function (cdf) is

$$F(x) = 1-\left(\frac{x^2}{\nu}+1\right)^{\frac{1-\nu}{2}}. \qquad (2)$$

where, $\nu > 0$ is the degree of freedom

3. Upper record values from the size-biased student's t distribution

Let $X_{U(1)}, X_{U(2)}, \cdots, X_{u(n)}$ be the upper record values from a sequence of $\{X_i\}$, identically independently distributed from the size-biased student's t distribution. As the probability density function of upper record value $X_{U(n)}$ is defined by

$$f_n(x) = \frac{[R(x)]^{n-1}}{\Gamma n} f(x) \qquad , -\infty \le x \le \infty \qquad (3)$$

where, $R(x) = -\ln \bar{F}(x)$ $\quad ,0 < \bar{F}(x) < 1$

as, $\bar{F}(x) = 1 - F(x)$

The nth records value $X_{U(n)}$ from the size-biased student's t distribution is defined by

$$f_n(x) = \frac{(\nu-1)}{\nu\Gamma n} x \left(\frac{x^2}{\nu}+1\right)^{-\left(\frac{1+\nu}{2}\right)} \left[-\left(\frac{1-\nu}{2}\right)\ln\left(\frac{x^2}{\nu}+1\right)\right]^{n-1}, \qquad x>0; \nu>0. \qquad (4)$$

The cumulative distribution function $F_n(x)$ of the record values from the size-biased student's t distribution is

$$F_n(x) = \frac{1}{\Gamma n}\gamma(n,\omega), \qquad (5)$$

Where, $\gamma(n,s) = \int_0^s u^{n-1} e^{-u} du$ is lower incomplete gamma function and $s = -\left(\frac{1-\nu}{2}\right)\ln\left(\frac{t^2}{\nu}+1\right)$.

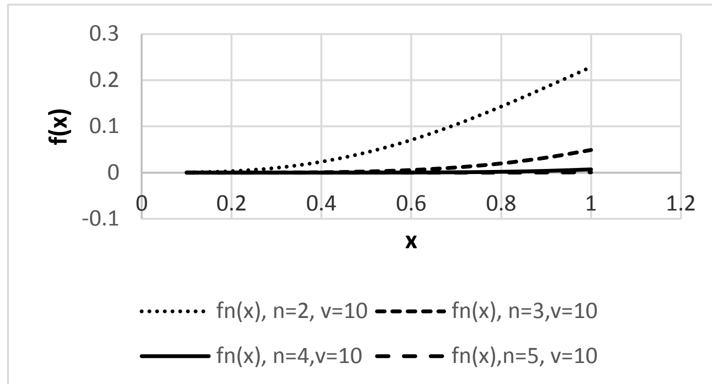

Fig.1 pdf graph for n = 2,3,4,5 and $\nu = 10$

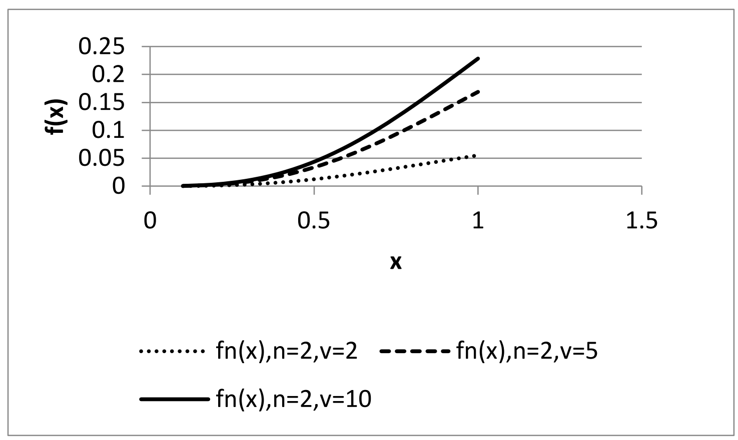

Fig.1 pdf graph for n = 2 and $\nu = 2,5,10$

The possible shapes of the pdf of upper record values from the size-biased student's t distribution are provided for different values of ν and n. The effect of ν can be easily seen from these graphs. The above graphs show that the distribution upper record values from the size-biased student's t is skewed with longer and heavier left tail.

4. Distributional properties of record values from the size-biased student's t distribution

In this section we derive the moments, survival function, hazard rate and entropy of the record values $\{X_{(n)}\}, n \geq 1$, obtained from the size-biased student's t distribution.

4.1 Moments

The rth moments of the upper record values by using the pdf in Eq. (4), when the parent distribution is size-biased student's t distribution

$$E\left(X_{(n)}^{r}\right)=\mu'_{r,(n)}=\frac{(\nu-1)}{\nu\Gamma n}\int_{0}^{\infty}x^{r+1}\left(\frac{x^2}{\nu}+1\right)^{-\left(\frac{1+\nu}{2}\right)}\left[-\left(\frac{1-\nu}{2}\right)\ln\left(\frac{x^2}{\nu}+1\right)\right]^{n-1}d$$

$$E\left(X_{(n)}^{r}\right)=\nu^{r/2}\sum_{k=0}^{\infty}\binom{r/2}{k}(-1)^{\frac{r}{2}-k}\left(\frac{1-\nu}{1-\nu+2k}\right)^{n} \quad (6)$$

Mean and variance of the upper record values from the size-biased student's t distribution are, respectively,

$$E\left(X_{(n)}\right)=\sqrt{\nu}\sum_{k=0}^{\infty}\binom{1/2}{k}(-1)^{\frac{1}{2}-k}\left(\frac{1-\nu}{1-\nu+2k}\right)^{n} \quad (7)$$

$$Var\left(X_{(n)}\right)=\nu\left[\sum_{k=0}^{\infty}\binom{1}{k}(-1)^{1-k}\left(\frac{1-\nu}{1-\nu+2k}\right)^{n}-\left(\sum_{k=0}^{\infty}\binom{1/2}{k}(-1)^{\frac{1}{2}-k}\left(\frac{1-\nu}{1-\nu+2k}\right)^{n}\right)^{2}\right] \quad (8)$$

4.2 Survival and hazard functions

The survival and hazard functions of the upper record values from the size-biased student's t distribution are given as, respectively

$$S_n(x)=\frac{\Gamma n-\gamma(n,s)}{\Gamma n} \quad (9)$$

and

$$h_n(x)=\frac{(\nu-1)}{\nu(\Gamma n-\gamma(n,s))}x\left(\frac{x^2}{\nu}+1\right)^{-\left(\frac{1+\nu}{2}\right)}\left[-\left(\frac{1-\nu}{2}\right)\ln\left(\frac{x^2}{\nu}+1\right)\right]^{n-1} \quad (10)$$

Where $\gamma(n,s)=\int_{0}^{s}u^{n-1}e^{-u}du$ is lower incomplete gamma function and $s=-\left(\frac{1-\nu}{2}\right)\ln\left(\frac{x^2}{\nu}+1\right)$.

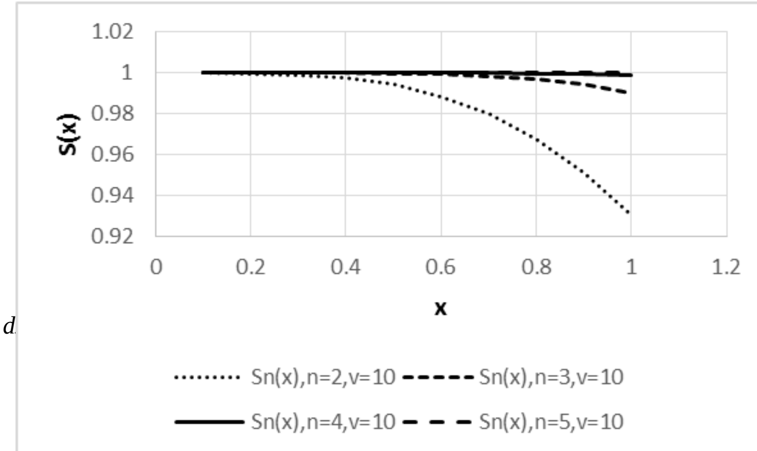

Fig. 3 Survival function for n = 2,3,4,5 and $\nu=10$

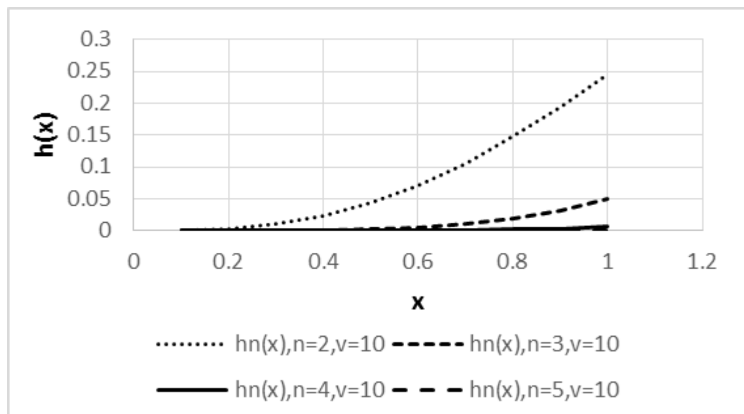

Fig. 4 Hazard function for n = 2,3,4,5 and $\nu=10$

The above graphs are provided with n = 2,3,4,5 and with $\nu=10\,\&\,1.5$. Fig 3 shows that survival function is monotonically decreasing. Fig. 4 shows that the hazard function is decreasing with n increasing. And the size-biased student's t distribution has a bathtub hazard function.

4.3 Entropy

Entropy provides an outstanding tool to compute the amount of information or uncertainty contained in a random observation concerning its parental distribution. A large value of entropy infers the more uncertainty in the data. The entropy of the record values from the size-biased student's t distribution is defined as

$$H_n(x) = -\ln\Gamma n - \frac{(n-1)}{\Gamma n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{i=0}^{\infty}(-1)^{k-i}\binom{k}{i}\Gamma(n+i) - \ln\left(\frac{\nu-1}{\sqrt{\nu}}\right) - \frac{1}{\Gamma n}\psi(t) - \frac{n(1+\nu)}{(1-\nu)}, \tag{11}$$

where $\psi(t) = \int_0^{\infty}\frac{1}{2}\ln\left(e^{-2t/(1-\nu)} - 1\right)t^{n-1}e^{-t}dt$

5. Confidence interval

Teimouri and Gupta (2012) considered $X_{U(n)}$ to be the upper record value from a family of the densities with cdf $F(.)$. Then a $100(1-\alpha)$ % confidence interval for the upper and lower record statistics are as follows.

$$\left[F^{-1}\left(1-\exp\left(-G_n^{-1}(g/2)\right)\right), F^{-1}\left(1-\exp\left(-G_n^{-1}(1-g/2)\right)\right)\right] \tag{12}$$

$$\left[F^{-1}\left(\exp\left(-G_n^{-1}(1-\gamma/2)\right)\right), F^{-1}\left(\exp\left(-G_n^{-1}(\gamma/2)\right)\right)\right] \tag{13}$$

Theorem: Let $X_{U(n)}$ and $X_{L(n)}$ be, respectively the nth upper and lower record statistics from size-biased student's t distribution. Then a $100(1-\alpha)$ % confidence interval for upper and lower record statistics are given as follows. Where $0 < \alpha < 1$.

$$\left[\left\{\nu\left(\exp\left(-G_n^{-1}(g/2)\right)\right)^{\frac{2}{1-\nu}} + \nu\right\}^{\frac{1}{2}}, \left\{\nu\left(\exp\left(-G_n^{-1}(1-g/2)\right)\right)^{\frac{2}{1-\nu}} + \nu\right\}^{\frac{1}{2}}\right] \tag{14}$$

$$\left[\left\{\nu\left(1-\exp\left(-G_n^{-1}(1-\gamma/2)\right)\right)^{\frac{2}{1-\nu}} + \nu\right\}^{\frac{1}{2}}, \left\{\nu\left(1-\exp\left(-G_n^{-1}(\gamma/2)\right)\right)^{\frac{2}{1-\nu}} + \nu\right\}^{\frac{1}{2}}\right] \tag{15}$$

Proof: The pdf of $X_{U(n)}$ and $X_{L(n)}$ are given in Ahsanullah (1987, 2004):

$$f_{X_{U(n)}} = \frac{[-\ln(1-F(x))]^{n-1}}{\Gamma(n)}f(x), \tag{16}$$

and

$$f_{X_{L(n)}} = \frac{[-\ln F(x)]^{n-1}}{\Gamma(n)}f(x). \tag{17}$$

where, $n \geq 1$, let $t = -\ln(1-F(x))$.

$$F_{X_{U(n)}} = \int_0^{y'}\frac{t^{n-1}e^{-t}}{\Gamma(n)}dt = F_{G_n}(y'),$$

where $y' = -\ln(1-F(y))$, and a continuous random variable G_n is said to have a gamma distribution.

$$F_{X_{U(n)}} = P(G_n < -\ln(1-F(y)))$$

$$= P\left(G_n < -\ln\left(\frac{y^2}{\nu}+1\right)^{\frac{1-\nu}{2}}\right)$$

By simplifying it we get the expression in Eq. (14)

Now let $t = -\ln F(x)$

$$F_{X_{U(n)}} = P(G_n > -\ln F(y))$$

$$= P\left(G_n < -\ln\left(1-\left(\frac{y^2}{\nu}+1\right)^{\frac{1-\nu}{2}}\right)\right)$$

By simplifying it we get the expression in Eq. (15).

4. Conclusions

In this paper, we discussed the distribution of upper record values from the size-biased student's t distribution. We derived the rth moments of the upper record values from the size-biased student's t distribution. The pdf, cdf, moments, survival function, hazard function, entropy etc. have been derived along with necessary graphs to describe the shapes of the respective functions. The associated graphs of pdf show that with the distribution of size-biased student's t record values is skewed with longer left tail, and graphs show the survival function is monotonically decreasing. Plots of hazard function shows increasing trend with n = 2, while decreasing function when n increase and has a bathtub shape. We also derive confidence intervals for ν to estimate the value of ν. We hope this paper will contribute a valuable contribution for the enhancement of research in the theory of record values.

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