An Algorithm for Fibonacci Labeling of a Tree

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Abstract

Abstract. A simple graph G is graceful if there exists a graceful labeling of the vertices of G. A graph G with e edges has a graceful labeling if there exists an injective function $\lambda: V(G) \rightarrow \{0,1,\ldots..e\}$ such that $|\lambda(x) - \lambda(y)|$ is distinct and nonzero for all $xy \in E(G)$. Graceful labeling is one of the best known labeling methods of graphs. In this work, we try to give an algorithm for Fibonacci labeling.

Keywords: Graceful labeling, Graceful graph, Fibonacci labeling.

1. Introduction

Let G = (V, E) be an undirected finite graph without loops or multiple edges. All parameters in this paper are positive integers. A graceful labeling (or β-labeling) of a graph G = (V, E) with n vertices and m edges is a one-to-one mapping Ψ of the vertex set V (G) into the set {0, 1, 2, . . . ,m} with this property: if we define, for any edge $e = (u, v) \in E(G)$, the value $\Omega(e) = |\Psi(u) - \Psi(v)|$, then Ω is a one-to-one mapping of the set E(G) onto the set {1, 2, . . . ,m}. A graph is called graceful if it has a graceful labeling. The concept of a graceful labeling has been introduced by Rosa as a means of attacking the famous conjecture of Ringel that K_{2n+1} can be decomposed into 2n + 1 sub graphs that are all isomorphic to a given tree with n edges.

Trees have been worked on mostly for Graceful and Harmonious labelings as discussed in Gallian (1998). The results obtained are often for special classes of trees and are thus limited and not generalized. Among the trees known to be Graceful are caterpillars given by Rosa (1967). Trees with at most 4 end vertices have been shown to be Graceful by Huang, Kotzig and Rosa (1982) and Jin, Meng and Wang (1993). Aldred and McKay (1992) have shown Gracefulness of trees with at most 27 vertices. Caterpillars are shown to be Sequential by Grace (1983) and Felicitous by Shee and Lee (1989). Aldred and McKay (1992) used a computer to show that all trees with at most 26 vertices are Harmonious. Whereas, Krishnaa (2001) has given a generalized computerized solution and algorithms for the major graph labeling schemes for smaller graphs. This comprehensive solution checks the existence of any of the major labeling schemes namely Harmonious, Felicitous, Sequential, Graceful. Magic and Antimagic labeling schemes, for an arbitrary graph, producing great number of labelings when they exist. Graham and Sloane (1980) showed caterpillars to be Harmonious. Bermond (1979) conjectured that lobsters (tree with the property that the removal of the endpoints leaves a caterpillar) and a number of authors have worked on special cases of this including Ng (1986) among others. Gallian (1998) has suggested for lobsters to be studied for being Harmonious as well. More specialized results of trees are contained in (Bermond, 1979; Cahit, 1989; Bloom, 1979;Koh et al., 1979) among others. Therefore, the results are not at all generalized and so restricted often to just one kind of tree and they render inapplicable for another type of tree. Graham and Sloane (1980) conjectured that trees are Harmonious and Ringel– Kotzig conjectured trees to be graceful.

Theorem 1
All trees on at most 27 vertices are graceful.

Before describing the method by which the above result is obtained, we introduce a similar problem of graph labeling to which we have been able to apply the same method.

Graham and Sloan made the following definition. An harmonious labeling of a given a graph G with q edges is an assignment to each vertex x ϵ V (G), a distinct element (x), of the group of integers modulo q, so that when each edge xy is labeled (x) + (y), the edge labels are all distinct. In the case of a tree with q edges, one element of the group is assigned to two vertices while all others are used precisely once. A graph which admits an harmonious labeling is said to be harmonious.

In the same paper it is verified that all trees on at most 10 vertices are harmonious and it is conjectured that all trees are harmonious. Employing a computer search similar to that used to find graceful labeling for trees.

2. Graceful graphs

Graceful labeling of trees may be viewed as a specialized sub-problem of 'graceful graph labeling.' Since graphs frequently have more edges than vertices, a graph with v vertices and e edges may have its vertices labeled from the set [0..e]. The edges must then be uniquely labeled with the integers [1..e] (Rosa 1967) (Figure 2.1).
(To be consistent with this, the vertex labels on a graceful tree should be the set [0..n- 1], instead of [1..n].
However, the latter usage has become common and will be used here.)

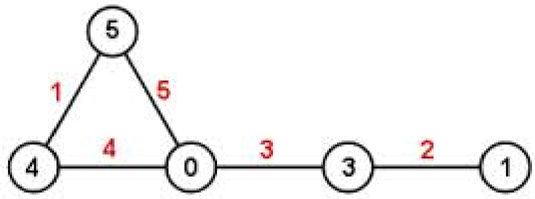

Figure. 2.1 Example of a gracefully labeled graph

Not all graphs can be gracefully labeled. Several classes of graph have been proven to be never graceful. Rosa has shown that if every vertex in a graph with e edges has even degree, and e mod 4 ∈□ [1,2], then the graph can never be gracefully labelled (Rosa 1967). A graph with these properties is in Figure 2.2.

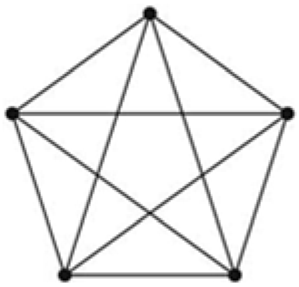

Figure. 2.2 A graph that cannot be gracefully labeled

3. Fibonacci numbers

The Fibonacci numbers $F_0, F_1, F_2 \dots \dots$ are defined by $F_0 = 0, F_1 = 1, F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

The Fibonacci numbers can be defined by the linear recurrence relation satisfying the following condition.

$$F_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F_{n-1} + F_{n-2}, & \text{if } n > 1 \end{cases}$$

Fibonacci graceful labeling

The function $f: V(G) \to \{0,1,2,\dots,F_q\}$ (where F_q is the q^{th} Fibonacci number) is said to be Fibonacci graceful if $f^*: E(G) \to \{F_1, F_2, \dots, F_q\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is bijective.

Deviating from the standard deviation of Fibonacci numbers it is assumed that $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$ which also avoid repetition of 1 as vertex(edge) label.

Some existing results

- K_n is Fibonacci graceful if and only if $n \le 3$.
- If G is Eulerian and Fibonacci graceful then $\equiv 0(mod3)$.
- Every path P_n of length n is Fibonacci graceful.
- ${P_n}^2$ is a Fibonacci graceful graph.
- Caterpillars are Fibonacci graceful.

Theorem 2

Trees are Fibonacci graceful.

Proof. Consider a vertex with minimum eccentricity as the root of tree T . Let this vertex be v.

Without loss of generality at each level of tree T we initiate the labeling from left to right.

Let $P^1, P^2, P^3, \dots \dots \dots . P^n$ be the children of v.

Define $f: V(T) \to \{0,1,2,\dots,F_q\}$ in the following manner.

$f(v) = 0 \, , f(P^1) = F_1$

Now if P^1_{1i} $(1 \le i \le t)$ are children of P^1 then $f(P^1_{1i}) = f(P^1) + F_{i+1}, \ (1 \le i \le t)$

If there are r vertices at level two of P^1 and out of these r vertices, r_1 be the children of P^1_{11} then label them as follows,

$f(P^1_{11i}) = f(P^1_{11}) + F_{t+1+i}\,, \qquad 1 \le i \le r_1$

Let there are r_2 vertices, which are children of P^1_{12} then label them as follows,

$f(P^1_{12i}) = f(P^1_{12}) + F_{t+1+r_1+i}\ , 1 \le i \le r_2$

Following the same procedure to label all the vertices of a subtree with root as P^1.

We can assign label to each vertex of the subtree with roots as $P^2, P^3, \dots \dots . . P^{n-1}$ and define $f(P^{i+1}) = F_{f_i+1}$, where F_{f_i} is the ${f_i}^{th}$ Fibonacci number assigned to the last edge of the tree rooted at P^i.

Now for the vertex P^n, Define $f(P^n) = F_q$

Let us denote P^n_{ij}, where i is the level of vertex and j is number of vertices at i^{th} level. At this stage one has to be cautious to avoid the repetition of vertex labels in right most branch. For that we first assign vertex label to that vertex which is adjacent to F_q and is a internal vertex of the path whose length is largest among all the paths whose origin is F_q(That is, F_q is a root). Without loss of generality we consider this path to be a left most path to F_q and continue label assignment from left to right as stated earlier.

If P^n_{1i} $(1 \le i \le s)$ be the children of P^n then define

$f(P^n_{1i}) = f(P^n) - F_{q-i}, \qquad 1 \le i \le s$

If there are P^n_{2i} $(1 \le i \le b)$ vertices at level two of P^n and out of these b vertices, b_1 be the children of P_{11} then label them follows.

$f(P^n_{2i}) = f(P^n_{11}) - F_{q-s-i}, \qquad 1 \le i \le b_1$

If there are b_2 vertices, which are children of P^n_{12} then label them as follows.

$f\left(P^n_{2(b_1+i)}\right) = f(P^n_{12}) - F_{q-S-b_1-i}\ , 1 \le i \le b_2$

We will also consider the situation when all the vertices of subtree rooted at F_q is having all the vertices of degree two after i^{th} level then we define labeling as follows.

$f(P^n_{(i+1)1}) =$
$f(P^n_{i1}) + (-1)^i F_{q-(labeled\ vertices\ in\ the\ branch\ rooted\ at\ P^n)}$

Continue this labeling scheme unless all the vertices of a subtree with root as P^n are labeled.

Thus we have labeled all the vertices at each level.

That is, T admits Fibonacci graceful labeling and accordingly trees are Fibonacci graceful graph.

Consider the tree with 12 edges then the Fibonacci graceful labeling is as follows.

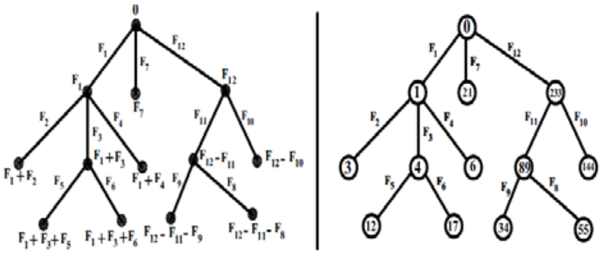

Figure. 2.3 Fibonacci graceful labelling of a tree

Algorithm for Fibonacci graceful labeling of a tree

1. Consider a vertex with minimum eccentricity as the root of tree T. Let this vertex be v.
2. Let $P^1, P^2, P^3, \ldots \ldots \ldots . P^n$ be the children of v, then $f(v) = 0\, , f(P^1) = F_1$
3. Introduce that if P^1_{1i} $(1 \le i \le t)$ are children of P^1 then $f(P^1_{1i}) = f(P^1) + F_{i+1},\ (1 \le i \le t)$
4. If there are r vertices, r_1 be the children of P^1_{11}, $f(P^1_{11i}) = f(P^1_{11}) + F_{t+1+i}\, , \quad 1 \le i \le r_1$
5. Let there be r_2 vertices, which are children of P^1_{12} , then $f(P^1_{12i}) = f(P^1_{12}) + F_{t+1+r_1+i}\ , 1 \le i \le r_2$
6. Follow the same to label all the vertices of a subtree with root as P^1.
7. Continue this labeling unless all the vertices of a subtree with root as P^n are labeled.
8. The resulting labeling is Fibonacci graceful graph.

5. Conclusion

This work has presented the generalized solutions to obtain the major labeling schemes for trees viz., Fibonacci graceful graphs. In labeling of the trees, the properties of its structure namely the p vertices and $(p - 1)$ edges, connectivity and acyclic have plus the crucial redrawing the tree as a bipartite graph led to the solutions. The resulting algorithm is not difficult to work with either.

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