# Application of Chromaticity for Cartesian Products

**Dr.B.R. Srinivas[1], A.Sri Krishna Chaitanya2**

[1]**Associate Professor of Mathematics, St.Marys College of Engineering &Technology, Chebrolu, Guntur (DT). AP.**

[2]**Associate Professor of Mathematics, Chebrolu Engineering College, Chebrolu, Guntur (DT), AP.**

**Abstract:**

This paper studies the Chromatic Number of Cartesian Products and Permutation Graphs. The main results of this paper are for every two graphs G and H, Chromatic Number of Cartesian Product G and H is the maximum of chromatic number Of G and Chromatic number of H and for every graph G and every Permutation Graph of G, the Chromatic number of Graph G is not more Than the Chromatic number of Permutation Graph and 4/3 of Chromatic Number of Graph G.

**Mathematics Subject Classification 2000**:68R10, 05CXX, 05C15, 05C30, 0 5C75

**Key words:** Chromatic number, Cartesian product of Graphs, Permutation Graph.

## §1.Introduction:

A proper vertex coloring of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. When it is understood that we are dealing with a proper vertex coloring, we ordinarily refer to this more simply as a coloring of G. While the colors used can be elements of any set, actual colors (such as red, blue, green, and yellow) are often chosen only when a small number of colors are being used; otherwise, positive integers (typically 1,2,………, k for some positive integer k) are commonly used for the colors. A reason for using positive integers as colors is that we are often interested in the number of colors being used. Thus, a )proper) coloring can be considered as a function c: V(G)→ N (where N is the set of positive integers) such that c(u) ≠c(v) if u and v are adjacent in G. if each color used is one of k given colors, then we refer to the coloring as a k-coloring. In a k-coloring, we may then assume that it is the colors 1,2,……….,k that are being used. While all k colors are typically used in a k-coloring of a graph, there are occasions when only some of the k colors are used.

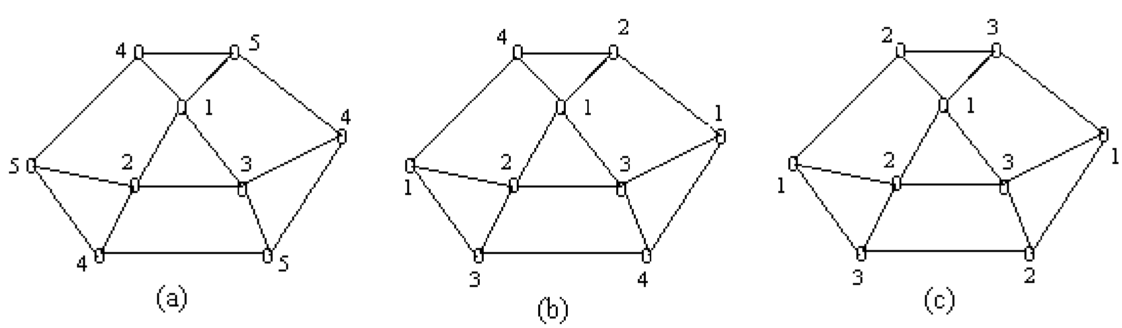
Suppose that c is a k-coloring of graph G, where each color is one of the integers 1,2,……,k as mentioned above. If $V_i$ ($1 \leq i \leq k$) is the set of vertices in G Vi is called a color class and the nonempty elements of $\{V_1, V_2, \ldots, V_k\}$ produce a partition of V(G). Because no two adjacent vertices of G are assigned the same color by c, each nonempty color class $V_i$ ($1 \leq i \leq k$) is an independent set of vertices of G.

**§1.1 Definition:** A graph G is k-colorable if there exists a coloring of G from a set of k colors. In other words, G is k-colorable if there exists a k-coloring of G. the minimum positive integer k for which G is k-colorable is the **chromatic number** of G and is denoted by $\chi(G)$.The chromatic number of a graph G is therefore the minimum number of independent sets into which V(G) can be partitioned. A graph G with chromatic number k is a k-chromatic graph. Therefore, if $\chi(G) = k$, then there exists a k-coloring of G but not a (k-1) coloring. In fact, a graph G is k-colorable if and only if $\chi(G) \leq k$. certainly, every graph of order n is n-colorable. Necessarily, if a k-coloring of a k-chromatic graph G is given, then all k colors must be used.

**§1.2Example:** Three different colorings of graph H are shown in Figure the coloring in Figure is 5-coloring, the coloring in Figure is a 4-coloring, and the coloring in Figure 6.1(c) is a 3-coloring. Because the order of G is 9, the graph H is k-colorable for every integer k with $3 \leq k \leq 9$. Since H is 3-colorable, $\chi$ (H) $\leq 3$. There is, however, no 2-coloring of H because H contains triangles and the three vertices of each triangle must be colored differently.

Therefore $\chi(H) \geq 3$ and so

$\chi(H)=3$.



The argument used to verify that the graph H of Figure has chromatic number 3 is a common one. In general, to show that some graph G has chromatic Number k, say, we need to show that there exists a k-coloring of G (and so $\chi(G) \leq k$) and to show that every coloring of G requires at lease k colors (and so $\chi(G) \geq k$).There is no general formula for the chromatic number of a graph. Consequently, we will often be concerned and must be content with (1) determining the chromatic number of some specific graphs of interest or of graphs belonging to some classes of interest and (2) determining upper and / or lower bounds for the chromatic number of a graph. Certainly, for every graph G of order n, $1 \leq \chi(G) \leq n$.

A rather obvious, but often useful according to M.Behzard [1] lower bound for the chromatic number of a graph involves the chromatic numbers of its sub graphs.

**§2 Corollary: If H is a sub graph of a graph G, then $\chi(H) \leq \chi(G)$**

**Proof:**

Suppose that $\chi(G) = k$. Then there exists a k-coloring c of G. since c assigns distinct colors to every two adjacent vertices of G, the coloring c also assigns distinct colors to every two adjacent vertices of H. therefore, H is k-colorable and so $\chi(H) \leq k = \chi(G)$.

Since the Cartesian product G x H of two graphs G and H contains sub graphs that are isomorphic to both G and H.

$$\chi(G \times H) > \max\{\chi(G), \chi(H)\}$$

**§3 Theorem: For every two graphs G and H, $\chi(G \times H) = \max\{\chi(G), \chi(H)\}$**

**Proof**:

From the concepts of chromatic number of a graph

$\chi(G \times H) > \max\{\chi(G), \chi(H)\}$.

Let $k = \max\{\chi(G), \chi(H)\}$.

Then there exist both a k-coloring $c^{l}$ of G and a k-coloring $c^{ll}$ of H.

we define a k-coloring c of G x H by assigning the vertex (u,v) of G x H the color c(u,v), where $0 \leq c(u,v) \leq k - 1$ and $c(u,v) \equiv (c^{l}(u) + c^{ll}(v)) \pmod{k}$.

To show that c is a proper coloring, let (x,y) be a vertex adjacent to (u,v) in G x H. Then either u = x and uy $\in$ E (H), (or) v = y & vx $\in$ E(G) say the former.

The (u,y) is adjacent to (u,v).

Hence $0 \leq c(u,y) \leq k-1$ and $c(u,y) \equiv (c^{l}(u) + c^{ll}(y)) \pmod{k}$.

Since uy $\in$ E (H), it follows that cll (v) $\neq c^{ll}(y) \pmod{k}$ and

$c(u,v) \equiv c^{l}(u) + c^{ll}(v) \neq c^{l}(u) + c^{ll}(y) \equiv c(u,y) \pmod{K}$.

Hence

$$\chi(G \times H) = \max\{\chi(G), \chi(H)\}$$

**§3.1 Corollary: For every nonempty graph G,**

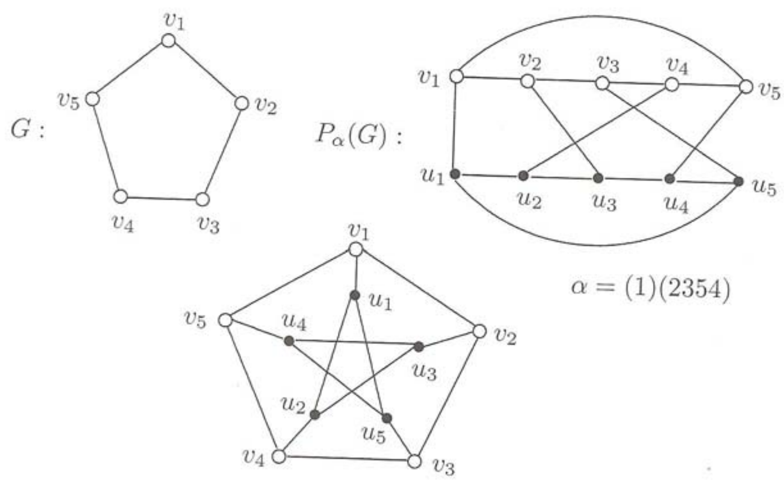
$$\chi(G \times K_2) = \chi(G).$$

The Cartesian product G x $K_2$ of a graph G and $K_2$ is a special case of a more general class of graphs. Let G be a graph with V (G) = $\{v_1, v_2, \ldots\ldots, v_n\}$ and let $\alpha$ be a permutation of the set S = {1,2,…,n}. by the **permutation graph** $P_\alpha$ (G) we mean the graph of order 2n obtained from two copies of G, where the second copy of G is denoted by $G^l$ and the vertex $v_i$ in G is denoted by $u_i$ in $G^l$ and $v_i$ is joined to vertex $u_{\alpha(i)}$ in $G^l$ .The edges $v_i u_{\alpha(i)}$ are called the **permutation edges** of $P_\alpha$ (G). Therefore, if $\alpha$ is the identity map on S, then $P_\alpha$ (G) = G x $K_2$. Since G is a sub graph of $P_\alpha$ (G) for every permutation $\alpha$ of s .It follows that $\chi(P_\alpha(G)) > \chi(G)$.

For example, consider the graph G = $C_5$ of Figure 2 and the permutation $\alpha$ = (1) (2354) of the set {1, 2, 3, 4,5}. Then the graph $P_\alpha$ ($C_5$) is also shown in Figure2 Redrawing $P_\alpha$ ($C_5$), we see that this is infact, the Petersen graph. Thus

$\chi(C_5) = \chi(P_\alpha(C_5)) = 3$.All four permutation graphs of C+5+ appear on the cover of the book Graph Theory by Frank Harary[5].

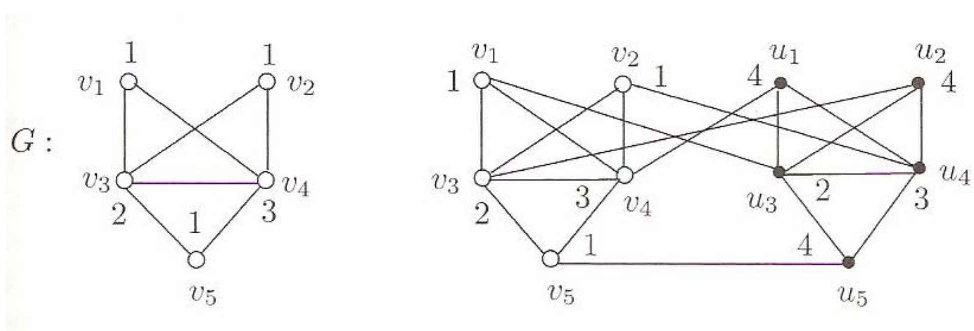
IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 2 Issue 1, January 2015.
www.ijiset.com
ISSN 2348 – 7968




2. The Petersen graph as a permutation graph

The examples we've seen thus far might suggest that if G is a nonempty graph with V(G) = $\{v_1, v_2, \ldots. v_n\}$ and α is a permutation on the set S = {1,2,…,n}, Then $\chi(G) = \chi(P_\alpha(G))$. This, however, is not the case.

Let G be the graph of order 5 shown in Figure 3 where V (G) = $\{v_1, v_2, \ldots. v_5\}$ and let α (1324) (5). The permutation graph Pα (G) is also shown in Figure 3




3. A permutation graph $P_\alpha(G)$ with $\chi(P\alpha(G)) > \chi(G)$

Certainly, χ (G) = 3 for the graph G of Figure 3. Therefore, $\chi(P\alpha(G)) \geq 3$ for the permutation graph $P_\alpha$ (G) of Figure We claim that $\chi(P\alpha(G)) > \chi(G)$. Suppose that $\chi(P\alpha(G)) = 3$. Then there exists a 3-coloring c of Pα (G). We may assume that $c(v_1) = c(v_2) = c(v_5) = 1$, $c(v_3) = 2$, and $c(v_4) = 3$. Since none of $u_3$, $u_4$, and $u_5$ can be colored 1, two of

ISSN 2348 – 7968

these vertices must be colored either 2 or 3. Since they are mutually adjacent, this is a contradiction. The 4-coloring of $P_\alpha$ (G) in Figure 3 shows that $\chi$ ($P_\alpha$ (G)) = 4. No permutation graph of the graph G of Figure 3 can have chromatic number greater than 4,however,according to the following theorem of Gary Chartrand and Joseph B.Frechen[3].

**§4 Theorem: For every graph G and every permutation graph $P_\alpha$ (G) of G,**
**$\chi$ (G) ≤ $\chi$ ($P_\alpha$ (G) ≤ [4/3 $\chi$(G) ]**

**Proof**:

Let G be a graph of order n and let $P_\alpha$ (G) be a permutation graph of G. Since G is a sub graph of $P_\alpha$ (G), is follows that $\chi(G) \leq \chi(P_\alpha (G))$. It remains therefore to establish the upper bound for $\chi(P_\alpha (G))$.

Suppose that $\chi$ (G) = k. if k =1, then $\chi$ ($P_\alpha$ (G)) = 2 = $[^{4k}/_3]$ Thus we may assume that k ≥ 2. Suppose that ∈ is the identity permutation on the set S = {1, 2,…,n}. Then $P_\alpha(G) = P_\in (G) = G \times K_2$. By Corollary 3.4 $\chi (P_\in (G)) = \chi (G) = K$.

We now show that for every permutation graph $P_\alpha$ (G) of G, there exists a $[^{4k}/_3]$-coloring of $P_\alpha$ (G). We begin with a k-coloring of G with the color classes $V_1$, $V_2$,……, $V_k$, where c(v) = i for each u ∈ $V_i$ for 1 ≤ i ≤ k and the same k-coloring of G' with color classes $V_1^1, V_2^1 \ldots, V_k^1$. We now consider two cases according to whether $[^{4k}/_3]$ is even or odd.

**Case 1**.$[^{4k}/_3]$ is even, say $[^{4k}/_3]$ = 2l for some positive integer l. we assign to each vertex of the set $V_i$', 1 ≤ i ≤ l, the color i ; while we assign to each vertex of the set $V_i$, 1 ≤ i ≤ l, the color i + l. For j = 1,2,…., k – l, we assign the color l + j to the vertices of $V_{l+j}$ that are not adjacent to any vertices of $V_j^1$ and assign the color 2l+1-j to the vertices of $V_{l+j}$ otherwise . In a similar manner, we assign the color
j = 1,2,…, k – l to the vertices of $V_{l+j}^1$ not adjacent to any vertices of $V_j$ and assign the color l + 1 – j to the vertices of $V_{l+j}^1$ otherwise. Since 4k/3 = 2l, it follows that 4k/3 ≤ 2l

and so $k \leq 3l/2$. hence there are sufficiently many colors for this coloring. Because this coloring of $P_\alpha$ (G) is proper coloring, $\chi$ ($P_\alpha$ (G) $\leq 2l = [^{4k}/_3]$

**Case 2**. $[^{4k}/_3]$ is odd, say $[^{4k}/_3] = 2l + 1$ for some positive integer l. we assign to each vertex of the set $V_i$ $1 \leq i \leq l$, the color i; while we assign the color $i + l + 1$ to the vertices of the set $V_i^l$, $1 \leq i \leq l$. For a vertex of $V_{l+j}$ for $2 \leq j \leq k - l$, we assign the color $l + j$ if it is not adjacent to any vertices of $V_{j-1}^l$ and assign the color
$2l + 3 - j$ otherwise. For a vertex $V_{l+j}^l$ for $1 \leq j \leq k - l$, we assign the color j if it is adjacent to no vertex of $V_j$ and assign the color $l + 2 - j$ otherwise. The fact that
$2 l + 1 \geq 4k/3$ give $k \leq (6l +3)/4$ and assures us that there are enough colors to accomplish the coloring. Since this coloring of ($P_\alpha$ (G)) is a proper coloring
$\chi$ ($P_\alpha$ (G)) $\leq 2l + 1 = [^{4n}/_3]$.

## References:

1. M .Behzard, Graphs and their chromatic numbers. Michigan state university (1965).

2. P.A. Catlin, A bound on the chromatic number of a graph. Discrete Math22. (1978) 81-83.

3. G.Chartand and J.B.Frechen, on the chromatic number of permutation graphs. Proof Techniques in graph theory (F.Harary, ed.), Academic press, New York (1969) 21-24.
4. F.M .Dong, K.M.Kho and K.L.Teo chromatic polynomials and chromaticity of Graphs. World scientific publishing (2005)

5 F.Harary, Graph Theory. Addison-Wesley, Reading, PA (1969).