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Graphs Embedded In Topological Spaces

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Abstract: We introduce a notion of graph homeomorphisms which uses the concept of embedded graphs into both orientable and non-orientable surfaces. It preserves the dimension of compact connectivity and classification of surfaces in classical topology. In this paper we shall give another sufficient condition for embedding of orientable and non orientable graphs to be uniquely and faithfully in compact connected surfaces.

Key words : Connected surfaces – faces - genus – orientable and non-orientable graphs - spanning tree.

Introduction :

A graph G is said to be **embedded in a surface F** if the vertices of G are distinct points in F and each edge of G is a simple arc in F connecting in F the vertices that are its end points in V such that its interior is disjoint from other vertices and edges. An embedding of a graph G in F is an isomorphism between G and a graph G that is embedded in F. If there is an embedding of G in F then it is said that G can be embedded in F.

In topological spaces X and Y , an embedding of X into Y is a continuous map $\varphi : X \to Y$ such that the restriction $\varphi : X \to \varphi(X)$ is homeomorphism , where $\varphi(X)$ has a **topology as a subspace** if Y embedding of a graph G.

Every topological map determines a 2-cell embedding of a graph X into a surface. Generally, an embedding $i : X \to S$ of a graph into a surface S is called a 2-cell embedding if each connectivity component of $S - i(X)$ is homeomorphic to an open disc. As mentioned above embeddings of a given graph X into orientable surfaces can be described by means of rotations permuting the set of darts of X. Our aim is to describe embeddings of graphs into both orientable and non-orientable surfaces in such a way that flags will not be referred. Recall that each vertex in an embedded graph corresponds to an orbit of ρ, τ , which can be viewed as an alternating cycle coloured by ρ and τ in $G(\lambda, \rho, \tau)$. Similarly, each edge is in a correspondence with an orbit of λ, τ , which can be viewed as an alternating cycle coloured by λ and τ in $G(\lambda, \rho, \tau)$. A (simple) cycle C in an embedded graph $X \to S$ will be called orientable if the subgraph of G induced by the edges of G of alternating cycles corresponding to vertices and edges of C is bipartite.

Otherwise C is called non-orientable. Choose a spanning tree T and set

$\xi(x) = 1 \in \{\pm 1, \cdot\}$ if x belongs to T . If x is a cotree dart we set $\xi(x) = 1 \in \{\pm 1, \cdot\}$ if the corresponding fundamental cycle in $T \cup \{x, x^{-1}\}$ is orientable, and $\xi(x) = -1 \in \{\pm 1, \cdot\}$ otherwise. This way a Cayley voltage space $(\{\pm 1, \cdot\}, \{\pm 1, \cdot\}, \xi)$ on X is determined. Clearly, $\xi(x) = 1$ for all $x \in D$ if and only if S is orientable

Given connected graph $X = (D, V ; I, L)$, an embedding scheme is a triple $(X; R, \xi)$, where R is a rotation and $\xi : D \to \{\pm 1, \cdot\}$ is a voltage assignment. We define the associated set F of closed walks given by the following rules:

(1) x and y are consecutive in $W \in F$ if $y = R\, \xi(x)\, (x-1)$,

(2) for each $x \in D$ either $\xi(x) = 1$ and both x, x^{-1} appear in the walks of F exactly once, or $\xi(x) = -1$ and x or x^{-1} (but not both) appear in the walks of F exactly twice.

Theorem 1:

A graph G has a non-separating embedding in an orientable surface S if and only if $\gamma(S) \geq (\beta(G) + \xi(G))/2$.

Proof:

Let S be an orientable surface of genus g and let G be a graph with k components G_1 , $G_2, \ldots, G_k$ such that

$$g \geq (\beta(G) + \xi(G))/2.$$

We show that G has a non-separating embedding in S. A non-separating embedding of G can be constructed as follows. Take any orientable surface R and for every component G_i of

G take a cellular embedding,

$$j_i : G_i \to S_i \text{ of } G_i \text{ in some orientable surface } S_i .$$

Let F_i be a closed collaring of j_i (G_i) in S_i , i.e., the closure of a "small" open neighbourhood of j_i (G_i) of which j_i (G_i) is a deformation retract. If the embedding ji has ri faces then F_i is a bordered surface with r_i boundary components containing j_i (G_i) in its interior. For each F_i and for each boundary component C of F_i remove an open disc DC from R and identify homeomorphically C with the boundary of DC in R. The identifications should be made in such a way that the resulting surface T will be orientable. Note that we thus obtain a non-separating embedding j of G in T ; we shall refer to j as the join of j_1 , j_2 , . . . , j_k by R.

Elementary computations show that if r = r_i is the total number of faces in the above cellular embeddings j_i , i = 1, 2, . . . , k, then

$$\gamma(T) = \gamma(R) + \gamma(S) + r - k .$$

In particular, choosing Si to have genus

$$\gamma(Si) = \gamma M (G_i) = (\beta(G_i) - \xi(G_i))/2,$$

R to have genus

$$\gamma(R) = g - (\beta(G) + \xi(G))/2$$

(which by our assumption is non-negative)

and using the fact that

$$r_i = \xi(G_i) + 1$$

we obtain that $\gamma(T) = g$.

Thus T is homeomorphic to S and the required non-separating embedding exists.

Conversely, assume that G is a graph having a non-separating embedding

$j : G \to S$ in an orientable surface S.

We show that

$$(\beta(G) + \xi(G))/2 \le \gamma(S).$$

Take a closed collaring F of j(G) in S. If G has k components G_1 , G_2 , . . . , G_k then F is the disjoint union of k bordered surfaces, each containing a component of j(G) in its interior. Let F_i be the component of F containing j(G_i). Then by capping each boundary component of F_i with a 2-cell we obtain a closed surface S_i and a cellular embedding $j_i : G_i \to S_i$, i = 1, 2, . . . , k. (This is the well-known "capping operation" of Youngs [7].)

Since S − j(G) is connected, so is S − Int(F) = H. Thus H is a bordered surface. Obviously, each boundary component of H is a boundary component of some F_i and vice versa. It follows that the number of boundary components of H is equal to the total number of

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faces in the embeddings j_i: $G_i \rightarrow S_i$, which we denote by r. By capping each boundary component of H with a 2-cell we obtain a closed surface R, and it is now clear that j is the join of $j_1, j_2, \ldots, j_k$ by R. Hence, employing (1) and the Euler formula for each j_i we have finally,

$$\gamma(S) \geq \gamma(R) + \gamma(S\ ) + r - k \geq 0 + (\beta(G) - r + k)/2 + r - k$$

$$= (\beta(G) + r - k)/2$$

$$\geq (\beta(G) + \xi(G))/2 .$$

This completes the proof.

Theorem 2:

Let X be a graph. Let $i : X \rightarrow S_g$, $j : X \rightarrow S_h$ be 2-cell embeddings of X into orientable surfaces of genera $0 \leq g < h$. Then for every integer γ, $g \leq \gamma \leq h$ there exists a 2-cell embedding of X into an orientable surface of genus γ.

Proof:

Given rotation R of darts of X we may form a new rotation R by switching the order of two consecutive darts in a cycle of R, so that a cycle $(x_1, x_2, \ldots, x_k)$ is replaced by $(x_1, x_2, \ldots, x_k)$.

We want to show that the number of faces of the new embedding of X described by R either remained the same, or it has changed by ±2. It is enough to see how the faces containing at the boundary the three consecutive pairs of darts (x^{-1}, x_1), (x^{-1}, x_2) and (x^{-1}, x_3) have been 1_{2k} changed. According to the distribution of the above pairs in faces of (X; R), there are six cases to discuss. The respective transformations of faces are indicated in the following table.

The embeddings i and j can be described by rotations P and Q, respectively. Clearly, the rotation Q can be obtained from P applying a finite number of elementary transpositions of consecutive elements. By the Euler-Poincare theorem in each step the genus of the underlying surface either remains the same, or it changes by ±. It follows that the integer interval [g, . . . , h] is covered by the genera of embeddings of X.

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Theorem 3:

Let X be a graph. Let i : X → Ng , j : X → Nh be 2-cell embeddings of X into non-orientable surfaces of genera $0 \leq g < h$. Then for every integer γ, $g \leq \gamma \leq h$ there exists a 2-cell embedding of X into an non-orientable surface of genus γ.

Proof:

We may assume that i be an embedding of X into a non-orientable surface with maximal number of faces. Then i can be described by an embedding scheme (X; R, ξ). Assume there are at least two faces in the embedding. Let e = $\{x, x^{-1}\}$ be an edge separating the two faces.

Set ξ (x) = −ξ(x) and ξ (x) = ξ(x), otherwise. Either the boundary walks of the two faces are of the form (Axy), (Bxz^{-1}), or of the form (Axy), (Bxz). In the new embedding the two faces transform to one faces bounded by $(Axz^{-1}Bxy^{-1})$ in the first case, and by (AxzBxy) in the second case. Repeating the above process we end with an one-face embedding of X. Since the closed walk W = (Axy) bounds a face,

ξ(W) = 1, thus

ξ (W) = −1.

Hence the non-orientability of the surface is preserved.

The proof is complete.

Theorem 4:

Let X be a connected graph without semi edges. Then minimum number of faces of a 2-cell embedding of X into an orientable surface is ζ(X) + 1. In particular, the maximum genus of X is equal (β(X) − ζ(X))/2.

Proof:

First we show that there exists an embedding of X with at most ζ(X) + 1 faces. We may assume that X is a simple graph, otherwise we can subdivide each edge by at most two vertices to make the graph simple. Let T be a spanning tree such that ζ(X, T) = ζ(X). First we construct a spherical one-face embedding of T . The co-tree edges in even connectivity components can be arranged into couples sharing a vertex vertex in common, and the co-tree edges in each odd connectivity component can be, except one, arranged into couples sharing at least one vertex. Denote by $R = \{e_1, e_2, \ldots, e_{\zeta(X)}\}$ the set of residual edges in the odd components.

Let $\{x, x^{-1}\},\{y, y^{-1}\}$ be a couple of incident co-tree with
$v = I(x^{-1}) = I(y^{-1})$, $u = I(x)$ and $w = I(y)$.

Since the embedding of T is one-face, vertices v, u and w appear at the boundary walk W of the face at least once. It follows that W can be written in the form

$$W = (ABC) = (vAuBwC).$$

We form a one-face embedding of $T \cup \{x, y\}$ transforming W onto $W = (Axy^{-1} Cx^{-1} By)$.

Since each dart is of the first type the embedding is orientable. Repeating the above procedure, we get a one-face embedding of $Y = X - R$. Let e_i, e_j be two residual edges incident with vertices u, v and z, w, respectively. We say that they are in a crossing position with respect to a one face embedding of Y if the boundary walk of the face can be expressed as $W = (ABCD)$ where $u = I(A)$, $z = I(B)$, $v = I(C)$ and $W = I(D)$. In such case we may form a one-face embedding of $Y \cup \{e_i, e_j\}$ by setting $W = (Ae_j De_i Ce^{-1} Be^{-1})$, here we have identified $e_i = \{e_i, e^{-1}\}$, $e_j = \{e_j, e_j^{-1}\}$ and we assume that $I(e_i) = z$ and $I(e_j) = u$. It follows that whenever we find a crossing pair of residual edges we may add the two edges to a one-face embedded graph and form a one-face embedding of the new graph. The procedure stops when each pair of the remaining residual edges is non-crossing. In such case we may draw them in the interior of the face forming a 2-cell embedding with at most $\zeta(X) + 1$ faces.

Let i be an embedding with f faces. We prove the inequality $f \geq \zeta(X) + 1$. There are edges $e_1, e_2, \ldots, e_{f-1}$ such that $Y = X - \{e_1, \ldots, e_{f-1}\}$ is a one-face embedded.

Since $\zeta(X) \leq \zeta(Y) + f - 1$, it is sufficient to prove $\zeta(Y) = 0$ for any one-face embeddable graph Y .

To see it we use induction by the number of edges of Y . The statement clearly holds for a trivial graph without edges. Since all the edges of Y are of the first kind, there is a dart x at the boundary walk W such that the distance of x and x−1 is minimal. It follows that W can be written in the form $W = (xAx^{-1} B)$. If A is an empty word then $e = \{x, x^{-1}\}$ is a pendant edge, removal of such an edge does not change $\zeta(Y)$ and so by induction

$\zeta(Y) = \zeta(Y - e) = 0.$

If $y \in \{x, x^{-1}\}$ is the first dart $y \in A$ then $y^{-1} \in B$.

It follows that W can be written as

$$W = (xyA_1 x^{-1} B_1 y^{-1} B_2),$$

where $A = xA_1$ and $B = B_1 y^{-1} B_2$.

Now $W' = (A_1 B_2 B_1)$ gives a one-face embedding of $Z = Y - \{x, x^{-1}, y, y^{-1}\}$.

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By induction hypothesis $\zeta(Y) = \zeta(Z) = 0$.

The expression $gMAX = (\beta(X) - \zeta(X))/2$ can be obtained from the Euler-Poincare formula inserting $f = \zeta(X) + 1$ for the number of faces.

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