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Recurrence Relations for Single and Product Moments of Ordinary Order Statistics from double truncated p^{th} Order Inverse Exponential Distribution

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Abstract

In this paper, we establish some recurrence relations satisfied by single and product inverse moments of ordinary order statistics from double truncated p^{th} order inverse exponential distribution. These recurrence relations are generalized versions of recurrence relations found by Joshi (1978, 1982), Balakrishnan and Rao (1998), Balakrishnan and Malik (1986a, b), Balakrishnan and Joshi (1984) and Mohie El-Din et al. (1997), Nain(2010a, b) and Arora and Sachdev (2015).

Keywords

Order statistics, generalized order statistics, record values, single inverse moment, product inverse moments, truncated distributions, exponential distribution, p^{th} order exponential distribution and p^{th} order inverse exponential distribution.

1. Introduction

Suppose $X_1, X_2, ..., X_n$ are elements of random sample of fixed size n drawn from a univariate continuous distribution having pdf f(x) and cdf F(x). Let the X_i's be arranged in order of magnitude from least to greatest and be denoted by $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$. These rearranged random variables are called order statistics. In particular, the r^{th} order statistic of a sample of size n is simply the r^{th} smallest observation in the sample and is denoted by $X_{r:n}$, where r is the index of the order statistic.

It may be mentioned that from the recurrence relations derived in this chapter, one can easily deduce the corresponding results for various other inverse distributions belongs to exponential family.

The pdf of $X_{r:n}$, $1 \le r \le n$, is given by

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r} f(x), \qquad -\infty < x < \infty, \tag{1.1}$$

and the joint probability density function of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$, $n \ge 2$, is given by

$$f_{X_{r:n},X_{s:n}}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left(F(x)\right)^{r-1}\left(F(y)-F(x)\right)^{s-r-1}$$

$$\times\left(1-F(y)\right)^{n-s} f(x) f(y), \qquad -\infty < x < y < \infty, \tag{1.2}$$

[cf. David and Nagaraja, 2003, p 12].

The moments of order statistics have generated considerable interest in the recent years. The expressions for several recurrence relations and identities satisfied by single as well as product moments of order statistics have been obtained by several authors in the past. These relations help in reducing the quantum of computations involved. Joshi (1978, 1982) established recurrence relations for exponential distribution with unit mean and were further extended by Balakrishnan and Joshi (1984) for doubly truncated exponential distribution. For linear-exponential distribution, Balakrishnan and Malik(1986) derived the similar type of relations which were extended to doubly truncated linear exponential distribution by Mohie El-Din et al. (1997) and Saran and Pushkarna (1999).

Nain (2010 a, b) obtained recurrence relations for ordinary order statistics and k^{th} record values from p^{th} order exponential and generalized weibull distributions, respectively.

In this paper, we have established recurrence relations for single and product inverse moments of p^{th} order inverse exponential distribution. This distribution generalizes inverse family of exponential distributions and has many applications in reliability analysis. The results so obtained are generalized versions of some of the recurrence relations obtained by Nain (2010) and Arora and Sachdev (2015).

2. p^{th} order Inverse Exponential Distribution

A random variable X is said to have p^{th} order inverse exponential distribution if its probability density function is of the form

$$f(x)=\left(\sum_{j=1}^{p}\frac{\alpha_j}{x^{j+1}}\right)e^{-\sum_{j=1}^{p}\frac{\alpha_j}{j\,x^j}},\ 0\le x<\infty, \tag{2.1}$$

where $\alpha_p > 0$ for some fixed positive integer p. The cumulative distribution function of the random variable X is given by

$$F(x)=e^{-\sum_{j=1}^{p}\frac{\alpha_j}{j x^j}},\ 0\le x<\infty. \tag{2.2}$$

and the functional relation between the pdf and the cdf is given by

$$f(x)=\left(\sum_{j=1}^{p}\frac{\alpha_j}{x^{j+1}}\right)F(x). \tag{2.3}$$

For different choices of $\alpha_j, j=0,1,2,\ldots, p$, the p^{th} order inverse exponential distribution represents many distributions , having applications in reliability analysis, survival models and many other mathematical models useful in health and care.

3. Doubly Truncated p^{th} Order Inverse Exponential Distribution

The doubly truncated p^{th} order inverse exponential distribution has probability density function

$$f(x)=\frac{1}{Q-P}\left(\sum_{j=1}^{p}\frac{\alpha_j}{x^{j+1}}\alpha_j\,x^j\right)e^{-\sum_{j=1}^{p}\frac{\alpha_j}{jx^j}},\ Q_1\le x<P_1, \tag{3.1}$$

where $Q = 1 - e^{-\sum_{j=1}^{p}\frac{\alpha_j}{jQ_1^j}}$ and $1 - P = e^{-\sum_{j=0}^{p}\frac{\alpha_j}{jP_1^j}}$ are the proportions of truncation on the left and right of the p^{th} order inverse exponential distribution (2.3). The proportions Q and P, with Q < P, are assumed to be known.

Assuming $P_2 = \frac{1-P}{P-Q}$ and $Q_2 = \frac{1-Q}{P-Q}$, one can easily observe the following relations:

- $Q_2 - P_2 = 1$, (3.2)
- $f(x) = (Q_2 - F(x))\left(-\sum_{j=1}^{p}\frac{\alpha_j}{x^{j+1}}\right)$, $Q_1 \le x \le P_1$, (3.3)
- $f(x) = (P_2 + (1 - F(x)))\left(-\sum_{j=1}^{p}\frac{\alpha_j}{x^{j+1}}\right)$, $Q_1 \le x \le P_1$, (3.4)

The mathematical forms connecting pdf and cdf, as given in (3.3) and (3.4), are called the characterizing differential equations for doubly truncated p^{th} order inverse exponential distribution and are very useful to derive recurrence relations for single and product moments of order statistics arising from doubly truncated p^{th} order inverse exponential distribution.

Notations

For n = 1, 2, 3,... , $u, v \in \{0, 1, 2, \ldots\}$ and $1 \le r < s \le n$,

(i) $\mu_{r:n}^{u} = E\left(\frac{1}{X_{r:n}^{u}}\right)$

(ii) $\mu_{r,s:n}^{u,v} = E\left(\frac{1}{X_{r:n}^{u} . X_{s:n}^{v}}\right)$.

4. Recurrence Relations for Single and Product Moments

Theorem 4.1 Let k = 0, 1, 2,... and $n \ge 1$.

(a). $$\mu_{1:n}^{k} = -n\sum_{j=1}^{p}\frac{\alpha_j}{k+j}\left(Q_2\, Q_1^{k+j} - P_2\mu_{1:n-1}^{k+j} + \mu_{1:n}^{k+j}\right). \quad (4.1)$$

(b) For $2 \le r \le n$,

$$\mu_{r:n}^{k} = -\sum_{j=1}^{p}\frac{\alpha_j}{k+j}\left[nP_2\left(\mu_{r:n-1}^{k+j} - \mu_{r-1:n-1}^{k+j}\right) + (n+1-r)\left(\mu_{r:n}^{k+j} - \mu_{r-1:n}^{k+j}\right)\right]. \quad (4.2)$$

Proof of (a). The k^{th} order moment (inverse moments) of $X_{1:n}$ is given by

$$\mu_{1:n}^{k} = n\int_{Q_1}^{P_1} x^{-k}(1-F(x))^{n-1} f(x)\,dx. \quad (4.3)$$

Substituting f(x) from (3.4), we get

$$\mu_{1:n}^{k} = -n\sum_{j=1}^{p}\alpha_j\int_{Q_1}^{P_1} x^{-(k+j+1)}(1-F(x))^{n-1}\left(P_2 + (1-F(x))\right)dx$$

$$= -n\sum_{j=1}^{p}\alpha_j\left\{P_2\,\omega(k,n-1) + \omega(k,n)\right\}, \quad (4.4)$$

where

$$\omega(k,n) = \int_{Q_1}^{P_1} x^{-(k+j+1)}(1-F(x))^{n}\,dx.$$

Integrating by parts, by treating $x^{-(k+j+1)}$ for integration and rest of the integrand for differentiation, we will get

$$\omega(k,n) = \frac{1}{k+j}\left(Q_1^{k+j} - \mu_{1:n}^{k+j}\right). \quad (4.5)$$

After substituting the values of $\omega(k,n-1)$ and $\omega(k,n)$ from (4.5) equations in equation (4.4), it will lead to (4.1).

Proof of (b). The kth order moment of $X_{r:n}$ is given by

$$\mu_{r:n}^{k} = \frac{n!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} x^{-k}\left(F(x)\right)^{r-1}(1-F(x))^{n-r} f(x)\,dx\,. \tag{4.6}$$

Substituting f(x) from (3.4), we have

$$\mu_{r:n}^{u} = \frac{n!}{(r-1)!(n-r)!}\sum_{j=1}^{p}\alpha_j\int_{Q_1}^{P_1} x^{-(k+j+1)}\left(F(x)\right)^{r-1}(1-F(x))^{n-r}\left(P_2+(1-F(x))\right)dx$$

$$= \frac{n!}{(r-1)!(n-r)!}\sum_{j=1}^{p}\alpha_j\left(P_2\,\psi(k,n)+\psi(k,n+1)\right), \tag{4.7}$$

where

$$\psi(k,n) = \int_{Q_1}^{P_1} x^{-(k+j+1)}\left((F(x))^{r-1}(1-F(x))^{n-r}\right)dx \qquad (4.8)$$

Integrating by parts, by treating $x^{-(k+j+1)}$ for integration and rest of the integrand for differentiation, we will get

$$\psi(k,n) = \frac{-1}{k+j}\left[n(n-r)\mu_{r:n-1}^{k+j} - n\mu_{r-1:n-1}^{k+j}\right] \tag{4.9}$$

On substituting $\psi(k,n)$ and $\psi(k,n+1)$ in (4.7) and then little simplifying, it leads to (4.2).

Theorem 2.3 Let $l, m \in \{\,0,1,2,...\}$ and n = 1,2,… .

(a) For $1 \le r \le$ n-1,

$$\mu_{r,r+1:n}^{l,m} = \sum_{j=1}^{p} \frac{\alpha_j}{m+j}\left[(n-r)\mu_{r:n}^{l+m+j} - (n-r)\mu_{r,r+1:n}^{l,m+j} + P_2\, n(\mu_{r:n-1}^{l+m+j} - \mu_{r,r+1:n-1}^{l,m+j})\right] \tag{4.10}$$

(b)For $1 \le r < s \le n$ and $s - r \ge 2$,

$$\mu_{r,s:n}^{l,m} = \sum_{j=1}^{p} \frac{\alpha_j}{m+j}\left[-nP_2\mu_{r,s:n-1}^{l,m+j} + P_2\mu_{r,s-1:n-1}^{l,m+j} - n\frac{n-1}{n-s}(\mu_{r,s:n-2}^{l,m+j} - \mu_{r,s-1:n-2}^{l,m+j})\right]. \tag{4.11}$$

Proof of (a). For $1 \le r \le n$, and on using expression for f(x, y) from (1.2), we get

$$\mu_{r,r+1:n}^{l,m} = \int_{Q_1}^{P_1}\int_{x}^{P_1} x^{-l} y^{-m} \frac{n!}{(r-1)!(n-r-1)!} F(x)^{r-1}(1-F(y))^{n-r-1} f(x) f(y)\, dx dy \tag{4.12}$$

Substituting

$$f(y) = \sum_{j=1}^{p} \frac{\alpha_j}{y^{j+1}}(P_2 + 1 - F(y)) \quad \text{from (3.4), we have}$$

$$\mu_{r,r+1:n}^{l,m} = \sum_{j=1}^{p} \alpha_j \int_{x=Q_1}^{P_1} x^{-l} \frac{n!}{(r-1)!(n-r-1)!} F(x)^{r-1} f(x) J(x) dx \tag{4.13}$$

Where

$$J(x) = P_2 \int_{y=x}^{P_1} y^{-(m+j+1)}(1-F(y))^{n-r-1} dy + \int_{y=x}^{P_1} y^{-(m+j+1)}(1-F(y))^{n-r} dy \tag{4.14}$$

$$= P_2 \int_{x}^{P_1} y^{-(m+j+1)}(1-F(y))^{n-r-1} dy + \int_{x}^{P_1} y^{-(m+j+1)}(1-F(y))^{n-r} dy \tag{4.15}$$

$$= P_2\omega(n-1) + \omega(n) \tag{4.16}$$

where

$$\omega(n) = \int_{x}^{P_1} y^{-(m+j+1)}(1-F(y))^{n-r} dy \tag{4.17}$$

Integrating $\omega(n)$ by parts, by treating $y^{-(m+j+1)}$ for integration and rest of the integrand for differentiation and substituting it in (4.16) and later, value of $J(x)$ in (4.13) we get(4.10)

Proof of (b). For $1 \le r < s \le n$, $s - r \ge 2$ and on using expression for f(y) from (3.4), we get

$$\mu_{r,s:n}^{l,m} = \int_{Q_1}^{P_1}\int_{x}^{P_1} x^{-l} y^{-m} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(x)^{r-1}$$

$$\times (F(y)-F(x))^{s-r-1}(1-F(y))^{n-s} f(x) f(y) dy dx \tag{4.18}$$

$$= \sum_{j=1}^{p} \alpha_j \int_{x=Q_1}^{P_1} x^{-l} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(x)^{r-1} J(x) f(x) dx \tag{4.19}$$

$$= P_2 \omega(n) + \omega(n-1) \tag{4.20}$$

were

$$J(x) = \int_{x}^{P_1} y^{-(m+j+1)} (1-F(y))^{s-r-1}(1-F(y))^{n-s}(P_2 + 1 - F(y)) dy \tag{4.21}$$

and

$$\omega(n) = \int_{x}^{P_1} y^{-(m+j+1)} (F(y)-F(x))^{s-r-1}(1-F(y))^{n-s} dy . \tag{4.22}$$

Again, Integrating $\omega(n)$ by parts, by treating $(F(y)-F(x))^{s-r-1}(1-F(y))^{n-s}$ for differentiation and rest of the integrand for integration and substituting it in (4.20) and later, value of J(x) in (4.21), we get(4.11).

4. Conclusion

In the study presented above, we demonstrate the recurrence relations for single and product moments of ordinary order statistics arising from p^{th} order inverse exponential distribution. These results generalize the corresponding results of Nain (2010) and Arora and Sachdev (2015).

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