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Some Common Fixed Point Theorems for Occasionally Weakly Compatible Mappings in Complex Valued Metric Space

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ABSTRACT

We proved some common fixed point theorems for occasionally weakly compatible mappings in complex valued metric space.

Keywords: Complex valued metric space, occasionally weakly compatible mappings, common fixed point theorem.

Mathematics subject classification: 47H10, 54H25.

1. Introduction

The famous Banach contraction principle states that if (X, d) is a complete metric space and $T: X \to X$ *is* a contraction mapping i.e., $d(Tx, Ty) \leq k\, d(x, y)$ for all $x, y \in X$, where k is a nonnegative number such that $k < 1$, then T has a unique fixed point. This result is one of the cornerstones in the development of nonlinear analysis.

Azam et al. [1] introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings involving rational expressions. Subsequently many authors have studied the existence and uniqueness of the fixed points and common fixed points of self mapping in view of contrasting contractive conditions.

The study of fixed point theorems, involving four single-valued maps, began with the assumption that all of the maps are commuted. Sessa [6] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [3] and then pairwise weakly compatible maps [4]. Jungck and Rhoades [5] proved some common fixed point theorems on the concept of occasionally weakly compatible maps.

Many researchers have obtained several fixed point theorems in complex valued metric spaces. We prove some common fixed point theorems for occasionally weakly compatible mappings in complex valued metric space.

2. Preliminaries

Let C be the set of complex numbers and let z1 , z2 $\in$C. Define a partial order $\leq$ on C as follows:
z1$\leq$z2 if and only if Re(z1)$\leq$Re(z2), Im(z1)$\leq$Im(z2). It follows that z1$\leq$z2 if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \leq z_2$ if one of (i),(ii) and (iii) is satisfied and we will write $z_1 < z_2$ if only (iii) is satisfied.

Definition 2.1. Let X be a non-empty set. Suppose that the mapping $d: X \times X \to C$ satisfies:

(a) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(c) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.
Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 2.2. Let $X = \mathbb{C}$. Define a mapping $d: X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ik}\,|z_1 - z_2|$,

where $k \in [0, \pi/2]$. Then , (X,d) is called a complex valued metric space.

Definition 2.3. Let f and g be self-maps on a set X, if $w = fx = gx$ for some x in X, then x is called coincidence point of f and g, w is called a point of coincidence of f and g.

Definition 2.4. Let f and g be two self-maps defined on a set X, then f and g are said to be weakly compatible if they commute at coincidence points.

Definition 2.5. Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

A. Al-Thagafi and Naseer Shahzad [2] shown that occasionally weakly is weakly compatible but converse is not true.

Example 2.6. Let R be the usual metric space. Define S, T: R $\to$R by Sx = 2x and Tx = x^2 for all $x \in R$. Then Sx = Tx for x = 0,2 but ST0 =TS0, and ST2 $\neq$ TS2. S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.7. Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

3. Main Results

Theorem 3.1. Let (X,d) be a complex valued metric space. Let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc such that

$$d^p(Ax, By) \leq$$

$$\varphi(ad^p(Sx,Ty) + (1-a)\max\{\alpha d^p(Ax,Sx), \beta d^p(By,Ty), d^{p/2}(Ax,Sx)d^{p/2}(Ax,Ty),$$

$$d^{p/2}(Ax,Ty)d^{p/2}(By,Sx), \tfrac{1}{2}[d^p(Ax,Sx) + d^p(By,Ty)]\}) \qquad (3.1)$$

for all $x, y \in X$, where $0 < a \leq 1$, $0 < \alpha, \beta \leq 1, p \geq 1$ and $\varphi: R^+ \to R^+$ such that φ is upper semi continuous, non decreasing and $\varphi(t) < t$ for any $t > 0$. Then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (3.1)

$$d^p(Ax, By) \leq$$

$$\varphi(ad^p(Ax,By) + (1-a)\max\{\alpha d^p(Ax,Ax), \beta d^p(By,By), d^{p/2}(Ax,Ax)d^{p/2}(Ax,By),$$

$d^{p/2}(Ax,By)d^{p/2}(By,Ax),\frac{1}{2}[d^p(Ax,Ax)+d^p(By,By)]\})$

$\leq \varphi(ad^p(Ax,By)+(1-a)\max\{0,0,0,d^p(Ax,By),0\})$

$\leq \varphi(ad^p(Ax,By)+(1-a)\max\{0,0,0,d^p(Ax,By),0\})$

$\leq \varphi\big(ad^p(Ax,By)+(1-a)\,d^p(Ax,By)\big)$

$\leq \varphi(d^p(Ax,By))$

$\leq d^p(Ax,By)$

a contradiction, therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is another point z such that $Az = Sz$ then by (3.1) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.7 w is the only common fixed point of A & S i.e. $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$d^p(w,z) = d^p(Aw,Bz)$

$\leq \varphi(ad^p(Sw,Tz)+(1-a)\max\{\alpha d^p(Aw,Sw),\beta d^p(Bz,Tz),$

$d^{p/2}(Aw,Sw)d^{p/2}(Aw,Tz),d^{p/2}(Aw,Tz)d^{p/2}(Bz,Sw),\frac{1}{2}[d^p(Aw,Sw)+d^p(Bz,Tz)]\})$

$\leq \varphi(ad^p(Sw,Tz)+(1-a)\max\{0,0,0,d^{p/2}(Aw,Tz)d^{p/2}(Bz,Sw),0\})$

$\leq \varphi(ad^p(Aw,Bz)+(1-a)\,d^p(Aw,Bz))$

$\leq \varphi(d^p(w,z))$

$= d^p(w,z)$

a contradiction, therefore we have $z = w$ by lemma 2.7, z is the unique common fixed point of A,B,S & T.

Theorem 3.2. Let (X,d) be a complex valued metric space. Let A,B,S and T be self-mappings of X. Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc such that

$$d(Ax,By) \leq \varphi(ad(Sx,Ty)+(1-a)\max\{\alpha d(Ax,Sx),\beta d(By,Ty),d^{1/2}(Ax,Sx)d^{1/2}(Ax,Ty),$$
$$d^{1/2}(Ax,Ty)d^{1/2}(By,Sx),\frac{1}{2}[d(Ax,Sx)+d(By,Ty)]\}) \quad (3.2)$$

for all $x,y \in X$, where $0 < a \leq 1$, $0 < \alpha$, $\beta \leq 1$ and $\varphi: R^+ \to R^+$ such that φ is upper semi continuous, non decreasing and $\varphi(t) < t$ for any $t > 0$. Then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A,B,S and T.

Proof: Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc, so there are points $x,y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (3.2)

$d(Ax,By) \leq \varphi(ad(Ax,By)+(1-a)\max\{\alpha d(Ax,Sx),\beta d(By,Ty),d^{1/2}(Ax,Sx)d^{1/2}(Ax,Ty),$

$d^{1/2}(Ax,Ty)d^{1/2}(By,Sx),\frac{1}{2}[d(Ax,Sx)+d(By,Ty)]\})$

$\leq \varphi(ad(Ax,By)+(1-a)\max\{\alpha d(Ax,Ax),\beta d(By,By),d^{1/2}(Ax,Ax)d^{1/2}(Ax,By),$

$d^{1/2}(Ax,By)d^{1/2}(By,Ax),\frac{1}{2}[d(Ax,Ax)+d(By,By)]\})$

$\leq \varphi(ad(Ax,By)+(1-a)\max\{0,0,0,d(Ax,By),0\})$

$\leq \varphi(ad(Ax, By) + (1-a)\, d(Ax, By))$

$\leq \varphi(ad(Ax, By) + (1-a)\, d(Ax, By))$

$\leq \varphi(d(Ax, By))$

a contradiction. Therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is another point z such that $Az = Sz$ then by (3.2) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.7 w is the only common fixed point of A & S i.e. $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$d(w, z) = d(Aw, Bz)$

$\leq \varphi(ad(Aw, Bz) + (1-a) \max\{\alpha d(Aw, Sw), \beta d(Bz, Tz), d^{1/2}(Aw, Sw) d^{1/2}(Aw, Tz),$

$d^{1/2}(Aw, Tz) d^{1/2}(Bz, Sw), \frac{1}{2}[d(Aw, Sw) + d(Bz, Tz)]\})$

$\leq \varphi(ad(w, z) + (1-a) \max\{\alpha d(w, w), \beta d(z, z), d^{1/2}(w, w) d^{1/2}(w, z),$

$d^{1/2}(w, z) d^{1/2}(z, w), \frac{1}{2}[d(w, w) + d(z, z)]\})$

$\leq \varphi(ad(w, z) + (1-a) \max\{0, 0, d(w, z), 0\})$

$\leq \varphi(ad(w, z) + (1-a)\, d(w, z))$

$\leq \varphi(d(w, z))$

$= d(w, z)$

a contradiction, since $(a + b + c) < 1$. Therefore we have $z = w$ by lemma 2.7 z is the unique common fixed point of A, B, S & T.

Theorem 3.3. Let (X, d) be a complex valued metric space. Let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc such that

$$d^p(Ax, By) \leq \varphi(ad^p(Sx, Ty) + (1-a) \max\left\{d^{\frac{p}{2}}(Ax, Ty) d^{\frac{p}{2}}(By, Sx)\right\}) \qquad (3.3)$$

for all $x, y \in X$, where $0 < a \leq 1$, $p \geq 1$ and $\varphi: R^+ \to R^+$ such that φ is upper semi continuous, non decreasing and $\varphi(t) < t$ for any $t > 0$. Then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (3.3)

$d^p(Ax, By) \leq \varphi(ad^p(Sx, Ty) + (1-a) \max\left\{d^{\frac{p}{2}}(Ax, Ty) d^{\frac{p}{2}}(By, Sx)\right\})$

$\leq \varphi(ad^p(Ax, By) + (1-a) \max\left\{d^{\frac{p}{2}}(Ax, By) d^{\frac{p}{2}}(By, Ax)\right\})$

$\leq \varphi(ad^p(Ax, By) + (1-a) \max\{d^p(Ax, By)\})$

$\leq \varphi(ad^p(Ax, By) + (1-a) d^p(Ax, By))$

$\leq \varphi(d^p(Ax, By))$

$= d^p(Ax, By)$

a contradiction . Therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is another point z such that $Az = Sz$ then by (3.3) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.7 w is the only common fixed point of A & S i.e. $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$d^p(w, z) = d^p(Aw, Bz)$

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$$\leq \varphi(ad^p(Sw,Tz)+(1-a)\max\{d^{\frac{p}{2}}(Aw,Tz)d^{\frac{p}{2}}(Bz,Sw)\})$$
$$\leq \varphi(ad^p(w,z)+(1-a)\max\{d^{\frac{p}{2}}(w,z)d^{\frac{p}{2}}(z,w)\})$$
$$\leq \varphi(ad^p(w,z)+(1-a)\max\{d^{\frac{p}{2}}(w,z)d^{\frac{p}{2}}(z,w)\})$$
$$\leq \varphi(ad^p(w,z)+(1-a)d^p(w,z))$$
$$\leq \varphi(d^p(w,z))$$
$$= d^p(w,z)$$

a contradiction, therefore we have $z = w$ by lemma 2.7 z is the unique common fixed point of A, B, S & T.

The next section of this paper presents some common fixed point theorems for occasionally weakly compatible mappings in complex valued metric space for six self maps.

Theorem 3.4. Let (X, d) be a complex valued metric space. Let AP, BQ, S and T be self-mappings of X. Let the pairs $\{AP, S\}$ and $\{BQ, T\}$ be owc such that

$$d(APx, BQy) \leq \psi\{d(Sx,Ty), d(Sx,APx), d(BQy,Ty), d(AP,Ty), d(BQy,Sx)\} \quad (3.4)$$

for all $x, y \in X$ & $\psi: [0,1]^5 \to [0,1]$ such that $\psi(t,0,0,t,t) < t$ for all $0 < t < 1$. Then there exist a unique point w ∈ X such that APw = Sw = w and a unique point z∈X such that BQz = Tz = z . Moreover, z = w, so that there is a unique common fixed point of AP, BQ, S and T. Furthermore, if the pairs (A,P) and (B,Q) are commuting pair of mappings then A, B, P, Q, S and T have a unique common fixed point.

Proof: Let the pairs $\{AP, S\}$ and $\{BQ, T\}$ be owc, so there are points $x, y \in X$ such that $APx = Sx$ and $BQy = Ty$. We claim that $APx = BQy$. If not, by inequality (3.4)

$$d(APx, BQy) \leq \psi\{d(Sx,Ty), d(Sx,APx), d(BQy,Ty), d(APx,Ty), d(BQy,Sx)\}$$
$$\leq \psi\{d(APx,BQy), d(APx,APx), d(BQy,BQy),$$
$$d(APx,BQy), d(BQy,APx)\}$$
$$\leq \psi\{d(APx,BQy), 0,0, d(APx,BQy), d(BQy,APx)\}$$
$$= d(APx, BQy)$$

a contradiction as $\psi: [0,1]^5 \to [0,1]$ such that $\psi(t,0,0,t,t) < t$ for all $0 < t < 1$. Therefore $APx = BQy$, i.e. $APx = Sx = BQy = Ty$. Suppose that there is another point z such that $APz = Sz$ then by (3.4) we have $APz = Sz = BQy = Ty$, so $APx = APz$ and $w = APx = Sx$ is the unique point of coincidence of AP and S. By Lemma 2.7 w is the only common fixed point of AP & S i.e. $w = APw = Sw$. Similarly there is a unique point $z \in X$ such that $z = BQz = Tz$.

Assume that $w \neq z$. We have

$$d(w,z) = d(APw, BQz)$$
$$\leq \psi\{d(Sw,Tz), d(Sw,APw), d(BQz,Tz), d(APw,Tz), d(BQz,Sw)\}$$
$$= d(w,z)$$

a contradiction as $\psi: [0,1]^5 \to [0,1]$ such that $\psi(t,0,0,t,t) < t$ for all $0 < t < 1$.Therefore we have $z = w$ by lemma 2.7 z is the unique common fixed point of AP, BQ, S & T. Finally, we need to show that w is only the common fixed point of mappings A, B, P, Q, S and T. If the pairs (A,P) and (B,Q) are commuting pairs, then for this we can write Aw = A(APw) = A(PAw) = AP(Aw). This implies that Aw = w. Also, Pw = P(APw) =

$PA(Pw) = AP(Pw)$. This implies that $Pw = w$. Similarly we have $Bw = w$ and $Qw = w$. Hence A, B, P, Q, S and T have a unique common fixed point.

Corollary 3.5. Let (X, d) be a complex valued metric space. Let AP, BQ, S and T be self-mappings of X. Let the pairs $\{AP, S\}$ and $\{BQ, T\}$ be owc such that

$$d(Ax, By) \leq \psi\{d(Sx, Ty), d(Ax, Ty), d(By, Sx)\} \quad (3.5)$$

for all $x, y \in X$ & $\psi: [0,1]^3 \to [0,1]$ such that $\psi(t, t, t) < t$ for all $0 < t < 1$. Then there exist a unique point w ∈ X such that APw = Sw = w and a unique point z∈X such that BQz = Tz = z . Moreover, z = w, so that there is a unique common fixed point of AP, BQ, S and T. Furthermore, if the pairs (A,P) and (B,Q) are commuting pair of mappings then A, B, P, Q, S and T have a unique common fixed point.

Proof: The proof follows from Theorem 3.4.

Corollary 3.6. Let (X, d) be a complex valued metric space. Let AP, BQ, S and T be self-mappings of X. Let the pairs $\{AP, S\}$ and $\{BQ, T\}$ be owc such that

$$d(Ax, By) \leq \psi max\{d(Sx, Ty), d(Sx, Ax), d(By, Ty), d(Ax, Ty), d(By, Sx)\} \quad (3.6)$$

for all $x, y \in X$ & $\psi: [0,1] \to [0,1]$ such that $\psi(t) < t$ for all $0 < t < 1$. Then there exist a unique point w∈X such that APw = Sw = w and a unique point z∈X such that BQz = Tz = z . Moreover, z = w, so that there is a unique common fixed point of AP, BQ, S and T. Furthermore, if the pairs (A,P) and (B,Q) are commuting pair of mappings then A, B, P, Q, S and T have a unique common fixed point.

Proof: The proof follows from Theorem 3.4.

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