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Solving Singular Partial Integro-Differential Equations Using Taylor Series

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Abstract

The aim of this study is to introduce a new technique to solve linear singular partial integro-differential equations (PIDEs) of first and second-order by using Taylor's series and convert the proposed PIDE to an partial differential equation. Solving this partial differential equation and applying the iteration method an exact solution of the problem is obtained. Some examples are presented in detail to show the accuracy and efficiency of this technique.

Keywords: *Partial integro-differential equations, Taylor's series, singular point*

1. Introduction

The theory and application of partial integro-differential equations (PIDEs) play an important role in the mathematical modeling of many fields: physical phenomena, biological models, chemical kinetics and engineering sciences in which it is necessary to take into account the effect of the real world problems.

The general form of linear PIDE is:

$$\sum_{i=0}^{n}\left(a_{1,i}(x,y)\frac{\partial^i f}{\partial x^i}(x,y) + a_{2,i}(x,y)\frac{\partial^i f}{\partial y^i}(x,y)\right) + \sum_{i,j=0}^{m} a_{3,i+j}(x,y)\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x,y) = u(x,y) + \int_a^x\int_b^y k(x,y,s,t) f(s,t)\,ds\,dt, \quad a < x \le c,\ b < y \le d, \tag{1}$$

where a, b, c and d are constants. $f(s,t)$ is the unknown function and $k(x,y,s,t)$ is the kernel of the integral equation. The functions $a_{1,i}(x,y), a_{2,i}(x,y), a_{3,i+j}(x,y), u(x,y)$ and $f(x,y)$ are usually assumed to be continuous on the intervals $[a,c]$ and $[b,d]$. Equations of this form are usually difficult to solve analytically so it is required to obtain an efficient approximate or numerical methods. These methods including Single-term Wash series method for Volterra integro-differential equations has been proposed by collocation method [4], Brunner applied a collecation-Sepehrian and Razzaghi [1], piecewise polynomials [2, 3], the spline type method to Volterra-Hammerstein integral equation as well as integro-di¤erential equations [5], the homotopy perturbation method (HPM) [6, 7], Haar wavelets [8], the wavelet-Galerkin method [9], the Tau method [10], the sinc-collocation method [11], the combined Laplace transform-Adomian decomposition method [12] to determine exact and approximate solutions, variational iterations method (VIM) [13] and Taylor polynomials[14].

The present work is motivated by the desire to obtain an exact solutions to first and second-order linear singular partial integro-differential equations, where the integrand is singular in the sense that its integral is continuous at the singular point, i.e. its kernel

$k(x,y,s,t) = \dfrac{1}{(x-t)^{\alpha}(y-s)^{\beta}}$ is singular as $t \to x$ and $s \to y$.

2. Solutions by Taylor's Series

We propose an exact solution for solving linear singular partial integro-differential equations.

The advantage of this method is that we remove the singularity of the kernel of first- and second-order linear singular partial integro-differential equations at $t = x$ and $s = y$ by judiciously applying Taylor's approximation and then transforming the given singular partial integro-differential equation into an partial differential equation.

2.1 First-order partial integro-differential equations

From Eq. (1), we can define the first-order singular partial integro-differential equation as:

$$h(x,y)\frac{\partial f(x,y)}{\partial x}+k(x,y)\frac{\partial f(x,y)}{\partial y}+q(x,y)f(x,y)=u(x,y)+\int_a^x\int_b^y \frac{f(s,t)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt, \quad \text{for } 0<\alpha<1 \text{ and } 0<\beta<1. \tag{2}$$

We can write

$$\begin{aligned}\int_a^x\int_b^y \frac{f(s,t)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt &= \int_a^x\int_b^y \frac{f(s,t)-f(x,y)+f(x,y)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt,\\ &= f(x,y)\int_a^x\int_b^y \frac{ds\,dt}{(x-t)^{\alpha}(y-s)^{\beta}}+\int_a^x\int_b^y \frac{f(s,t)-f(x,y)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt.\end{aligned} \tag{3}$$

For first-order partial differential equations, we use the following Taylor's approximation series of degree 1 of $f(s,t)$ about $t=x$ and $s=y$

$$f(s,t)\approx f(x,y)+(t-x)f_x(x,y)+(s-y)f_y(x,y),$$

or equivalently

$$f(s,t)-f(x,y)\approx (t-x)f_x(x,y)+(s-y)f_y(x,y), \tag{4}$$

substituting Eq: (4) into Eq: (3) we have

$$\int_a^x\int_b^y \frac{f(s,t)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt = f(x,y)\int_a^x\int_b^y \frac{ds\,dt}{(x-t)^{\alpha}(y-s)^{\beta}} - \int_a^x\int_b^y \frac{(x-t)f_x(x,y)\,ds\,dt}{(x-t)^{\alpha}(y-s)^{\beta}} - \int_a^x\int_b^y \frac{(y-s)f_y(x,y)\,ds\,dt}{(x-t)^{\alpha}(y-s)^{\beta}},$$

so

$$\int_a^x\int_b^y \frac{f(s,t)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt = \frac{(y-b)^{1-\beta}(x-a)^{1-\alpha}}{(\beta-1)(\alpha-1)}f(x,y) - \frac{(y-b)^{1-\beta}(x-a)^{2-\alpha}}{(1-\beta)(2-\alpha)}f_x(x,y) - \frac{(y-b)^{2-\beta}(x-a)^{1-\alpha}}{(2-\beta)(\alpha-1)}f_y(x,y),$$

thus Eq: (4) becomes:

$$h(x,y)\frac{\partial f(x,y)}{\partial x}+k(x,y)\frac{\partial f(x,y)}{\partial y}+q(x,y)f(x,y)=u(x,y)+\frac{(y-b)^{1-\beta}(x-a)^{1-\alpha}}{(\beta-1)(\alpha-1)}f(x,y)-\frac{(y-b)^{1-\beta}(x-a)^{2-\alpha}}{(\beta-1)(2-\alpha)}\frac{\partial f(x,y)}{\partial x}-\frac{(y-b)^{2-\beta}(x-a)^{1-\alpha}}{(2-\beta)(\alpha-1)}\frac{\partial f(x,y)}{\partial y}$$

or equivalently

$$\left[h(x,y)+\frac{(y-b)^{1-\beta}(x-a)^{2-\alpha}}{(1-\beta)(2-\alpha)}\right]\frac{\partial f(x,y)}{\partial x}+\left[k(x,y)+\frac{(y-b)^{2-\beta}(x-a)^{1-\alpha}}{(2-\beta)(\alpha-1)}\right]\frac{\partial f(x,y)}{\partial y}-u(x,y)=\left[\frac{(y-b)^{1-\beta}(x-a)^{1-\alpha}}{(\beta-1)(\alpha-1)}-q(x,y)\right]f(x,y).$$

Therefore

$$f(x,y)=-\frac{u(x,y)}{A}+\frac{B}{A}\frac{\partial f(x,y)}{\partial x}+\frac{C}{A}\frac{\partial f(x,y)}{\partial y}, \tag{5}$$

where

$$A=\frac{(y-b)^{1-\beta}(x-a)^{1-\alpha}}{(\beta-1)(\alpha-1)}-q(x,y)\neq 0,$$

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$$B = h(x,y) + \frac{(y-b)^{1-\beta}(x-a)^{2-\alpha}}{(1-\beta)(2-\alpha)},$$

$$C = k(x,y) + \frac{(y-b)^{2-\beta}(x-a)^{1-\alpha}}{(2-\beta)(1-\alpha)}.$$

The solution (5) in a series form can be written as

$$f(x,y) = \sum_{n=0}^{\infty} f_n(x,y) = -\frac{u(x,y)}{A} + \frac{B}{A}\frac{\partial}{\partial x}\sum_{n=0}^{\infty} f_n(x,y) + \frac{C}{A}\frac{\partial}{\partial y}\sum_{n=0}^{\infty} f_n(x,y),$$

and the recursion scheme

$$\begin{cases} f_0(x,y) = -\dfrac{u(x,y)}{A}, \\ f_n(x,y) = \dfrac{B}{A}\dfrac{\partial}{\partial x} f_{n-1}(x,y) + \dfrac{C}{A}\dfrac{\partial}{\partial y} f_{n-1}(x,y), \quad n \geq 1. \end{cases} \tag{6}$$

Example

If we consider Eq: (2) with $h(x,y) = k(x,y) = 1$, $q(x,y) = 0$, $\alpha = \beta = \frac{1}{2}$ and $u(x,y) = -4\sqrt{xy}$.

Then Eq: (2) becomes:

$$f_x(x,y) + f_y(x,y) = -4\sqrt{xy} + \int_0^x\int_0^y \frac{f(s,t)\,ds\,dt}{\sqrt{x-t}\sqrt{y-s}}$$

and we have

$$A = 4\sqrt{xy},\ B = 1 + \frac{4}{3}y^{\frac{1}{2}}x^{\frac{3}{2}} \ and\ C = 1 + \frac{4}{3}y^{\frac{3}{2}}x^{\frac{1}{2}}.$$

Then, the components $f_n(x,y)$ can be recursively by applying Eq: (6) as follows

$$\begin{cases} f_0(x,y) = -\dfrac{u(x,y)}{A} = 1, \\ f_n(x,y) = 0, \quad n \geq 1. \end{cases}$$

Thus the solution is

$$f(x,y) = \sum_{n=0}^{\infty} f_n(x,y) = 1.$$

Which is the exact solution.

2.2 Second-order partial integro-differential equations

Let us consider from Eq: (1) the second-order singular partial integro-differential equation, namely

$$a_1(x,y)\frac{\partial^2 f}{\partial x^2} + a_2(x,y)\frac{\partial^2 f}{\partial x \partial y} + a_3(x,y)\frac{\partial^2 f}{\partial y^2} + a_4(x,y)\frac{\partial f}{\partial x} + a_5(x,y)\frac{\partial f}{\partial y} + q(x,y)f(x,y) = u(x,y) + \int_a^x\int_b^y \frac{f(s,t)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt, \tag{7}$$

$$for\ 0 < \alpha < 1\ and\ 0 < \beta < 1.$$

We can write

$$\begin{aligned} \int_a^x\int_b^y \frac{f(s,t)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt &= \int_a^x\int_b^y \frac{f(s,t) - f(x,y) + f(x,y)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt, \\ &= f(x,y)\int_a^x\int_b^y \frac{ds\,dt}{(x-t)^{\alpha}(y-s)^{\beta}} + \int_a^x\int_b^y \frac{f(s,t) - f(x,y)}{(x-t)^{\alpha}(y-s)^{\beta}}\,ds\,dt. \end{aligned} \tag{8}$$

For second-order partial differential equations, we use the following Taylor's approximation series of degree 2 of $f(s,t)$ about $t = x$ and $s = y$

$$\begin{aligned} f(s,t) - f(x,y) \approx{}& (t-x)\frac{\partial f(x,y)}{\partial x} + (s-y)\frac{\partial f(x,y)}{\partial y} + \frac{(t-x)^2}{2}\frac{\partial^2 f(x,y)}{\partial x^2} \\ &+ (t-x)(s-y)\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{(s-y)^2}{2}\frac{\partial^2 f(x,y)}{\partial y^2} \end{aligned} \tag{9}$$

and then we have

$$f(x,y)\int_a^x\int_b^y \frac{ds\,dt}{(x-t)^{\alpha}(y-s)^{\beta}} = -\frac{(y-b)^{1-\beta}(x-a)^{1-\alpha}}{(\beta-1)(\alpha-1)} f(x,y), \quad (10)$$

$$\int_a^x\int_b^y \frac{(t-x) f_x(x,y)}{(x-t)^{\alpha}(y-s)^{\beta}}ds\,dt = -\frac{(y-b)^{1-\beta}(x-a)^{2-\alpha}}{(1-\beta)(2-\alpha)}\frac{\partial f(x,y)}{\partial x}, \quad (11)$$

$$\int_a^x\int_b^y \frac{(s-y) f_y(x,y)}{(x-t)^{\alpha}(y-s)^{\beta}}ds\,dt = -\frac{(y-b)^{2-\beta}(x-a)^{1-\alpha}}{(2-\beta)(\alpha-1)}\frac{\partial f(x,y)}{\partial y}, \quad (12)$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\int_a^x\int_b^y \frac{(t-x)^2}{(x-t)^{\alpha}(y-s)^{\beta}}ds\,dt = \frac{1}{2}\frac{(y-b)^{1-\beta}(x-a)^{3-\alpha}}{(1-\beta)(3-\alpha)}\frac{\partial^2 f}{\partial x^2}, \quad (13)$$

$$\frac{\partial^2 f}{\partial x\,\partial y}\int_a^x\int_b^y \frac{(t-x)(s-y)}{(x-t)^{\alpha}(y-s)^{\beta}}ds\,dt = \frac{(y-b)^{2-\beta}(x-a)^{2-\alpha}}{(2-\beta)(2-\alpha)}\frac{\partial^2 f}{\partial x\,\partial y} \quad (14)$$

and

$$\frac{1}{2}\frac{\partial^2 f}{\partial y^2}\int_a^x\int_b^y \frac{(s-y)^2}{(x-t)^{\alpha}(y-s)^{\beta}}ds\,dt = \frac{1}{2}\frac{(y-b)^{3-\beta}(x-a)^{1-\alpha}}{(3-\beta)(1-\alpha)}\frac{\partial^2 f}{\partial y^2}. \quad (15)$$

Then we can write Eq: (8) in the form

$$B\frac{\partial^2 f}{\partial x^2} + C\frac{\partial^2 f}{\partial x\,\partial y} + D\frac{\partial^2 f}{\partial y^2} + E\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial y} - u(x,y) = Af(x,y), \quad (16)$$

where that

$$A = \frac{(y-b)^{1-\beta}(x-a)^{1-\alpha}}{(\beta-1)(\alpha-1)} - q(x,y) \neq 0,$$

$$B = a_1(x,y) - \frac{1}{2}\frac{(y-b)^{1-\beta}(x-a)^{3-\alpha}}{(1-\beta)(3-\alpha)},$$

$$C = a_2(x,y) - \frac{(y-b)^{2-\beta}(x-a)^{2-\alpha}}{(2-\beta)(2-\alpha)},$$

$$D = a_3(x,y) - \frac{1}{2}\frac{(y-b)^{3-\beta}(x-a)^{1-\alpha}}{(3-\beta)(1-\alpha)},$$

$$E = a_4(x,y) + \frac{(y-b)^{1-\beta}(x-a)^{2-\alpha}}{(1-\beta)(2-\alpha)}$$

and

$$F = a_5(x,y) + \frac{(y-b)^{2-\beta}(x-a)^{1-\alpha}}{(2-\beta)(1-\alpha)}.$$

If we can write Eq: (16) in the form of a series solution

$$\begin{aligned} f(x,y) &= \sum_{n=0}^{\infty} f_n(x,y) \\ &= -\frac{u(x,y)}{A} + \frac{B}{A}\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} f_n(x,y) \\ &\quad + \frac{C}{A}\frac{\partial^2}{\partial x\,\partial y}\sum_{n=0}^{\infty} f_n(x,y), \\ &\quad + \frac{D}{A}\frac{\partial^2}{\partial y^2}\sum_{n=0}^{\infty} f_n(x,y), \\ &\quad + \frac{E}{A}\frac{\partial}{\partial x}\sum_{n=0}^{\infty} f_n(x,y), \\ &\quad + \frac{F}{A}\frac{\partial}{\partial y}\sum_{n=0}^{\infty} f_n(x,y), \end{aligned}$$

then, we can write the recursion scheme as follows:

$$\begin{cases} f_0(x,y) = -\dfrac{u(x,y)}{A}, \\ f_n(x,y) = \dfrac{B}{A}\dfrac{\partial^2}{\partial x^2} f_{n-1}(x,y) \\ \quad + \dfrac{C}{A}\dfrac{\partial^2}{\partial x\,\partial y} f_{n-1}(x,y) \\ \quad + \dfrac{D}{A}\dfrac{\partial^2}{\partial y^2} f_{n-1}(x,y) \\ \quad + \dfrac{E}{A}\dfrac{\partial}{\partial x} f_{n-1}(x,y) \\ \quad + \dfrac{F}{A}\dfrac{\partial}{\partial y} f_{n-1}(x,y), \quad n \geq 1. \end{cases} \quad (17)$$

Example

Consider the PIDE (7) with

$a_1 = a_3 = 1, \quad a_2 = a_4 = a_5 = 0, \quad q = 0$

$and \;\; u = -\frac{16}{9} x^{\frac{3}{2}} y^{\frac{3}{2}}.$

Therefore

$A = 4 y^{\frac{1}{2}} x^{\frac{1}{2}}, C = -\frac{4}{9} y^{\frac{3}{2}} x^{\frac{3}{2}}, E = \frac{4}{3} y^{\frac{1}{2}} x^{\frac{3}{2}}$

$and \;\; F = \frac{4}{3} y^{\frac{3}{2}} x^{\frac{1}{2}}.$

The components $f_n(x, y)$ can be recursively determined by applying Eq: (17) as follows

$$\begin{cases} f_0(x,y) &= \frac{4}{9} xy, \\ f_1(x,y) &= \frac{4}{9}\left(\frac{5}{9}\right) xy, \\ f_2(x,y) &= \frac{4}{9}\left(\frac{5}{9}\right)^2 xy, \\ f_3(x,y) &= \frac{4}{9}\left(\frac{5}{9}\right)^3 xy, \\ & \cdots \\ f_n(x,y) &= \frac{4}{9}\left(\frac{5}{9}\right)^n xy, \quad n \geq 1, \end{cases}$$

which we recognize as a geometric series. Thus

$$f(x, y) = \sum_{n=0}^{\infty} f_n(x, y) = xy.$$

Which is the exact solution.

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3. Conclusions

The new technique successfully uses to solve the first and second order partial integro-differential equations. The exact solution of PIDE after some steps of calculations has been done. This new technique is easy to implement and produces accurate results. Some other types of PIDE and these equations can be used in modeling real life phenomena.

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