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Special Pythagorean Triangles and Triangular Numbers

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Abstract: *Finding out the Special Pythagorean Triangles, where their perimeters are Triangular numbers, is the main objective of this paper. Cases, when one leg and a hypotenuse are consecutive, are also discussed. A few interesting results are observed. Various 3D graphs of corresponding Pythagorean triplets are plotted using software Mathematica.*

Key words: Pythagorean Triangles, Triangular Numbers.
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1. Introduction

Pythagorean Theorem continues to inspire mathematicians all over the world. The problems related to it keep on fascinating all those who love numbers and hence continues to enrich Mathematics. Triangular numbers played very important role in Pythagorean theory of numbers. Special Pythagorean Triangles are generated by Gopalan & Vijyalakshmi [1] and Gopalan and Devibala [2]. Integral solutions of ternary quadratic equation are given by Gopalan, Somnath & Vanitha [3] and by Gopalan & Kalinga Rani [4]. Rana and Darbari [5] obtained special Pythagorean Triangles, with their legs to be consecutive, in terms of Triangular Numbers. An attempt has been made to find special Pythagorean triangles with their perimeters as Triangular Numbers.

2. Method of Analysis:

The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2 \tag{2.1}$$

is given by [6]

$$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2 \tag{2.2}$$

for some integers ***m, n*** of opposite parity such that ***m*** > ***n*** > **0** and (*m*, *n*) = 1.

2.1 Perimeter is a Triangular number:

Definition 2.1: A natural number *p* is called a Triangular number if it can be written in the form

$$\frac{\beta(\beta+1)}{2}, \beta \in \mathrm{N}$$

If the perimeter of the Pythagorean Triangle (X, Y, Z) is Triangular number *p*, then

$$X + Y + Z = \frac{\beta(\beta+1)}{2} = p \tag{2.3}$$

By virtue of equation (2.2), equation (2.3) becomes

$$2m^2 + 2mn = \frac{\beta(\beta+1)}{2}, \beta \in \mathrm{N}$$

$$\text{Or, } 2m(m+n) = \frac{\beta(\beta+1)}{2} \tag{2.4}$$

$$\Rightarrow \beta = \frac{(-1+\sqrt{1+16\mathrm{m}^2+16mn})}{2} \text{ and}$$

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$$\beta = \frac{(-1-\sqrt{1+16m^2+16mn})}{2}$$

Discarding the second value of β as it is negative, we get

$$\beta = \frac{(-1+\sqrt{1+16m^2+16mn})}{2} \tag{2.5}$$

Solving equation (2.5) using *Mathematica* for 0 < m < 1000, 0 < n < 1000, 0 < β < 100000, 451 special Pythagorean Triangles are obtained with their perimeters as Triangular Numbers.
The following tables give 10 Primitive Pythagorean Triangles with their perimeter as triangular numbers:

S.N.	**M**	n	β	**X**	**Y**	**Z**	**X^2**	**Y^2**	**$X^2 + Y^2 = Z^2$**	X + Y + Z = β (β +1)/2
1	15	11	39	104	330	346	10816	108900	119716	780
2	21	17	56	152	714	730	23104	509796	532900	1596
3	22	13	55	315	572	653	99225	327184	426409	1540
4	33	25	87	464	1650	1714	215296	2722500	2937796	3828
5	35	16	84	969	1120	1481	938961	1254400	2193361	3570
6	39	31	104	560	2418	2482	313600	5846724	6160324	5460
7	40	17	95	1311	1360	1889	1718721	1849600	3568321	4560
8	42	23	104	1235	1932	2293	1525225	3732624	5257849	5460
9	51	19	119	2240	1938	2962	5017600	3755844	8773444	7140
10	52	47	143	495	4888	4913	245025	23892544	24137569	10296

Table 2.1: Primitive Pythagorean Triangles (X, Y, Z) with X + Y + Z = β (β + 1)/2

2.2 Hypotenuse and one leg are consecutive:

Now, if one leg and hypotenuse are consecutive, in such cases,

$$m = n + 1 \tag{2.6}$$

This gives equation (2.4) as

$$(2n+1)(2n+2) = \frac{\beta(\beta+1)}{2}$$

$$\Rightarrow \beta = \frac{-1 \pm \sqrt{32n^2 + 48n + 17)}}{2} \tag{2.7}$$

Solving equation (2.7) by using the software *Mathematica* for n < 10^{24} and for β < 10^{26}, it was found there was no solution!

2.3 Two legs are consecutive:

In the Primitive solutions of Pythagorean triangle, one of the legs is even and the other is odd. Y = 2*mn* is obviously even. If two legs are consecutive then,

$$X - Y = \pm 1 \tag{2.8}$$

Case1: Let X= Y + 1. Then by equation (2.3), we have

$$2Y + 1 + Z = \frac{\beta(\beta+1)}{2} = p$$

$$\Rightarrow m^2 + n^2 + 4mn + 1 = \frac{\beta(\beta+1)}{2} \tag{2.9}$$

Solving equation (2.9) using software *Mathematica* for $m < 10^{24}$, $n < 10^{24}$ and $\beta < 10^{30}$, we again get no solution!

Case2: Let Y= X + 1. Then by equation (2.3), we have

$$2X + 1 + Z = \frac{\beta(\beta+1)}{2} = p$$

$$\Rightarrow 3m^2 - n^2 + 1 = \frac{\beta(\beta+1)}{2} \tag{2.10}$$

Again solving equation (2.10) using software *Mathematica* for $m < 10^{24}$, $n < 10^{24}$ and $\beta < 10^{30}$, we do not get any solution!

3. 3D Plot:

As seen in section 2.1, for 0 < m < 1000, 0 < n < 1000, 0 < β < 100000, 451 special Pythagorean Triangles are obtained with their perimeters as Triangular Numbers. Plotting these Pythagorean Triplets (X, Y, Z) as ListPointPlot3D using software *Mathematica*, we get the following graph (Figure 2.1).

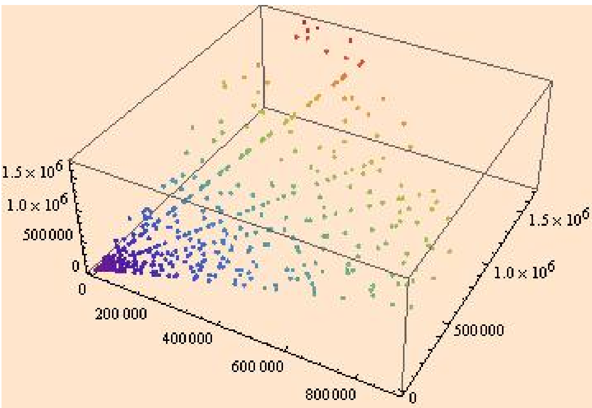

Figure 2.1:ListPlot3D of Special Pythagorean Triplets

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4. Observations and conclusion:

We observe that

1. $X + Y + Z = 0 \pmod 2$.
2. $Y + Z - X = 0 \pmod 2$.
3. $(X + Y + Z)(X + Y - Z) = 0 \pmod 8$.
4. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.
5. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z)$.

In conclusion, other special Pythagorean Triangle can be found which satisfy the conditions other than discussed in the above problem.

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