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# Solution of Integro-differential Equation of the second order with the operators

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## Abstract.

In this paper, we study the existence, uniqueness and stability solution of integro-differential equations of second order with the operators by using both method Picard approximation and Banach fixed point theorem.These investigations lead us to improving and extending the above method. . Thus the integro-differential equations of second order with the operators are more general and detailed than those introduced by Butris.

**Keywords**.Existence, uniqueness and stability solution, nonlinear system, integro-differential equations ,Picard approximation , Banach theorem.

## I. Introduction

The periodic solutions for some nonlinear differential equations and integro-differential equations have been used to study many problems for example see [1,2,4, 6, 8 ].

Butris [3 ]has been used the above methods to consider the following problem:-

$$\frac{dx}{dt} = f\left(t, x, Ax, \int_0^{h(t)} g\big(s, x(s), Bx(s)\big)\, ds\right)$$

where $f(t, x, y, z)$ *and* $g(t, x, w)$ are continuous vector functions which are defined on the domains :

$$\left.\begin{array}{l}(t,x,y,z)\in R^1\times D\times D_1\times D_2=(-\infty,\infty)\times D\times D_1\times D_2\\ (t,x,w)\in R^1\times D\times D^*=(-\infty,\infty)\times D\times D^*\end{array}\right\}$$

where $x \in D \subset R^n$ , $D$ is closed and bounded domain subset of Euclidean space $R^n$ and $D_1$ , $D_2$ , $D^*$ are bounded domains subset of Euclidean space $R^m$ .

In this paper, we study the existence , uniqueness and also stability solution of integro-differential equations of the second order with the operators by using both method Picard approximation and Banach fixed point theorem which are given in [ 5 ] and [ 7 ] respectively and the problem which are studying is the following form:-

$$\frac{d^2x}{dt^2} = f\left(t, x, \dot{x}, Ax, A\dot{x}, \int_0^{h(t)} g(s, x, \dot{x}, Bx, B\dot{x})\, ds\right) \quad \ldots (P)$$

where $f(t,x,\dot{x},y,\dot{y},z)$ and $g(t,x,\dot{x},w,\dot{w})$ are continuous vector functions which are defined on the domains :-

$$\left.\begin{array}{l}(t,x,\dot{x},y,\dot{y},z)\in R^1\times D\times D_1\times D_2\times D_3\times D_4=(-\infty,\infty)\times D\times D_1\times D_2\times D_3\times D_4\\ (t,x,\dot{x},w,\dot{w})\in R^1\times D\times D^*\times D^{**}=(-\infty,\infty)\times D\times D^*\times D^{**}\end{array}\right\}$$

…(1)

where $x \in D \subset R^n$ , $D$ is closed and bounded domain subset of Euclidean space $R^n$ and $D_1$ , $D_2$ , $D_3$ , $D_4$ $D^*$, $D^{**}$ are bounded domains subset of Euclidean space $R^m$ .

Suppose that in the domain (1) the vector functions $f\ (t,x,\dot{x},y,\dot{y},z)$, $g(t,x,\dot{x},w,\dot{w})$ and the operators A and B satisfy the following inequalities:-

$$\|f\ (t,x,\dot{x},y,\dot{y},z)\| \le M\ ,\ \|g(t,x,\dot{x},w,\dot{w})\| \le N \quad \cdots (2)$$

$$\left\|f\left(t,x_1,\dot{x}_1,y_1,\dot{y}_1,z_1\right)-f(t,x_2,\dot{x}_2,y_2,\dot{y}_2,z_2)\right\| \le K[\ \|x_1-x_2\| + \|\dot{x}_1-\dot{x}_2\| + \|y_1-y_2\| + \left\|\dot{y}_1-\dot{y}_2\right\| + \|z_1-z_2\|\ ] \quad \cdots (3)$$

$$\|g(t,x_1,\dot{x}_1,w_1,\dot{w}_1)-g(t,x_2,\dot{x}_2,w_2,\dot{w}_2)\| \le P\ [\ \|x_1-x_2\| + \|\dot{x}_1-\dot{x}_2\| +$$

$$\|w_1 - w_2\| + \|\dot{w}_1 - \dot{w}_2\| ] \cdots (4)$$

$$\| h(t) \| \leq h < \infty \quad \ldots (5)$$

$$\|Ax_1 - Ax_2\| \leq Q_1\|x_1 - x_2\|$$

$$\|A\dot{x}_1 - A\dot{x}_2\| \leq Q_2\|\dot{x}_1 - \dot{x}_2\| \quad \ldots (6)$$

$$\|Bx_1 - Bx_2\| \leq Q_3\|x_1 - x_2\|$$

$$\|B\dot{x}_1 - B\dot{x}_2\| \leq Q_4\|\dot{x}_1 - \dot{x}_2\| \quad \ldots (8)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $\dot{x}, \dot{x}_1, \dot{x}_2 \in D_1$, $y, y_1, y_2\ D_2$, $y, \dot{y}_1, \dot{y}_2 \in D_3$ $z, z_1, z_2 \in D_4$, $w, w_1, w_2 \in D^*$, $\dot{w}, \dot{w}_1, \dot{w}_2 \in D^{**}$ .

where $M, N, h$ *and* $K, L, Q_1, Q_2, Q_3, Q_4$ are a positive constants. We assume *that the operators A and B are defined in the class of continuous functions and map it into the class continuous functions.*

We define the non-empty sets as follows:-

$$\left.\begin{aligned} D_f &= D - M\frac{T^2}{2} \\ D_{1f} &= D_1 - MT \\ D_{2f} &= D_2 - Q_1 M\frac{T^2}{2} \\ D_{3f} &= D_3 - Q_2 MT \\ D_{4f} &= D_4 - hPMT\left[ (1 + Q_4) + \left(\frac{1 + Q_3}{2}\right) \right] \end{aligned}\right\} \quad \ldots (9)$$

We consider the matrix $\Lambda = \begin{pmatrix} \frac{'H_1 T^2}{2} & \frac{'H_2 T^2}{2} \\ 'H_1 T & 'H_2 T \end{pmatrix}$

where $'H_1 = K(1 + +\mathrm{P\,h} + Q_3\mathrm{P\,h})$

$'H_2 = K(1 + Q_2 + \mathrm{P\,h} + Q_4\mathrm{P\,h})$

Furthermore, we assume that

the largest eigen value $\lambda_{\max}$ of the matix $\Lambda$ dose not exceed unity.

that is $\lambda_{\max}(\Lambda) = \frac{{'H}_1 T^2 + 2{'H}_2 T}{2} < 1$

…(10)

Define the sequences of functions $x(t, x_0)$ and $\dot{x}(t, x_0)$ on the domain

$(t, x_0, \dot{x}_0) \in R^1 \times D_f \times$

….(11)

by the following:-

$$x_{m+1}(t, x_0) = x_0 + \dot{x}_0 t + \int_0^t \int_0^t [\, f(s, x_m(s, x_0), \dot{x}_m(s, x_0), Ax_m(s, x_0), A\dot{x}_m(s, x_0), \int_0^{h(s)} g\,(\tau, x_m(\tau, x_0), \dot{x}_m(\tau, x_0), Bx_m(\tau, x_0), B\dot{x}_m(\tau, x_0) d\tau\,]\, ds\, ds \quad (12)$$

with $x_0(0, x_0) = x_0 + \dot{x}_0 t$ , $m$ =0,1,2,….

and

$$\dot{x}_{m+1}(t, x_0) = \dot{x}_0 + \int_0^t [\, f(s, x_m(s, x_0), \dot{x}_m(s, x_0), Ax_m(s, x_0), A\dot{x}_m(s, x_0), \int_0^{h(s)} g\,(\tau, x_m(\tau, x_0), \dot{x}_m(\tau, x_0), Bx_m(\tau, x_0), B\dot{x}_m(\tau, x_0) d\tau\,]\, ds \;\ldots\; (13)$$

with $x_0(0, x_0) = \dot{x}_0$ , $m = 0,1,2, \ldots$ .

***Definition 1.[5].*** *A continuous function $f$ satisfy a* ***Lipschitz condition*** *on the domain $G = \{(t, x): a \le t \le b, c \le x \le d\}$ in the variable $x$ on $G$ if for all $K > 0$ and $(t, x_1), (t, x_2) \in G$, such that $|f(t, x_1) - f(t, x_2)| \le K|x_1 - x_2|$.*

***Definition 2.[7].*** *A solution $x(t)$ is said to be* ***stable*** *if for each $\varepsilon > 0$ ,*
*There exists a $\delta > 0$ such that any solution $\bar{x}(t)$ which satisfies*
$\|\bar{x}(t_0) - x(t_0)\| < \delta$ *for some $t_0$ , also satisfies*

$\|\bar{x}(t) - x(t)\| < \varepsilon \qquad for\ all\ t \geq t_0$ .

***Definition 3.[5].*** *Let (C [0, T] , ‖.‖ ) be a norm space if T maps into itself we say that T is a* ***contraction mapping*** *on C [0,T] if there exists* $\alpha \in R$ *with* $0 < \alpha < 1$ *such that* $\|Tx - Ty\| \leq \alpha\|x - y\|, (x, y) \in\ C\,[0, T]$.

***Theorem 1. [ 5 ] .****Let E be a Banach space , if T is a contraction mapping on E then T has one and only one fixed point in E .*

## II. Existence Solution of (P).

In this section, we prove the existence theorem for second integro-differential equation (P) by using the Picard approximation method.

**Theorem 2.** (**Existence theorem)** .Let the vector functions $f\ (t, x, \dot{x}, y, \dot{y}, z)$, $g(t, x, \dot{x}, w, \dot{w})$ be defined on the domain (1) continuous in $t, x, \dot{x}, y, \dot{y}, z, w, \dot{w}$ and satisfy the inequalities( 2) to (8)and the conditions(9),(10) . Then there exist a sequence of functions (12) convergent uniformly to the limit function$x_\infty(\mathrm{t}, x_0)$ , defined on the domain (11) and satisfy the following integral equations:-

$$x(\mathrm{t}, x_0) = \ x_0 + \dot{x}_0 t + \int_0^t \int_0^t [f(s, x(s, x_0), \dot{x}(s, x_0), Ax(s, x_0)\,, A\dot{x}(s, x_0),$$

$$\int_0^{h(t)} g\ (\tau, x(\tau, x_0), \dot{x}(\tau, x_0)\,, Bx(\tau, x_0), B\dot{x}(\tau, x_0)d\tau\ ]\ ds\ ds \qquad \text{... (15)}$$

which is a solution of (P).

**Proof .**

By mathematical induction we can prove that:-

$\|\dot{x}_m(\mathrm{t}, x_0) - \ \dot{x}_0\ \| \leq\ M\ T$ …

(16)

From (16) we obtain the estimate

$\|A\ \dot{x}_m(\mathrm{t}, x_0) - \ A\ \dot{x}_0\ \| \leq\ Q_2 M\ T$

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which gives $\dot{x}_m(\mathrm{t}, x_0) \in D_1$ , $A\dot{x}_m(\mathrm{t}, x_0) \in D_3$ for all $\mathrm{t} \in [0, T]$ and $\dot{x}_0 \in D_{1f}$ , $A\dot{x}_0(\mathrm{t}, x_0) \in D_{3f}$ .

And from the sequence of functions (12), for all $m$

$$\|x_m(\mathrm{t}, x_0) - x_0\| \leq M\,\frac{T^2}{2}. \qquad \ldots (17)$$

Also from (17) we have

$$\|Ax_m(\mathrm{t}, x_0) - A\,x_0\| \leq Q_1 M\,\frac{T^2}{2}.$$

That is $x_m(\mathrm{t}, x_0) \in D$ , $Ax_m(\mathrm{t}, x_0) \in D_2$ for all $\mathrm{t} \in [0, T]$ and $x_0 \in D_f$ , $Ax_0(\mathrm{t}, x_0) \in D_{2f}$ .

Also

$$\|z_1(\mathrm{t}, x_0) - z_0(\mathrm{t}, x_0)\| = \left\| \int_0^{h(t)} g(s, x_1(s, x_0),\ \dot{x}_1(s, x_0), Bx_1(s, x_0),) \right.$$

$$\left. , B\dot{x}_1(s, x_0))ds - \int_0^{h(t)} g(s, x_0,\ \dot{x}_0, Bx_0, B\dot{x}_0)ds \right\|$$

$$\leq \int_0^{h(t)} \Big\| P\,[\,\|x_1(s, x_0) - x_0\| + \|\dot{x}_1(s, x_0) - \dot{x}_0\| + \|Bx_1(s, x_0) - Bx_0\|$$

$$+ \left\|B^{\cdot} x_1(s, x_0) - B\dot{x}_0\right\|\,]\Big\|\ ds$$

$$\leq hPMT\,[\,(1+Q_4) + (\tfrac{1+Q_3}{2})\,],$$

i.e $z_1(\mathrm{t}, x_0) \in D_4$ for all $\mathrm{t} \in [0, T]$ and $z_0 \in D_{4f}$ .

Now, by mathematical induction we can prove that:-

$$\|z_m(\mathrm{t}, x_0) - z_0(\mathrm{t}, x_0)\| \leq hPMT\,[\,(1+Q_4) + (\tfrac{1+Q_3}{2})\,],$$

i.e $z_m(\mathrm{t}, x_0) \in D_4$ for all $\mathrm{t} \in [0, T]$ and $z_0 \in D_{4f}$ .

Next, we shall to prove that the sequence of functions (12) uniformly converges on the domain (1) .

By using the sequence of functions ( 12) when $m = 1$ , we get

$$\|\dot{x}_2(t,x_0) - \dot{x}_1(t,x_0)\|$$

$$\leq \int_0^t K\,[\,\|x_1(s,x_0) - x_0\| + \|\dot{x}_1(s,x_0) - \dot{x}_0\| + Q_1\|x_1(s,x_0) - x_0\| + Q_2\|\dot{x}_1(s,x_0) - \dot{x}_0\| + Ph\,(\|x_1(s,x_0) - x_0\| + \|\dot{x}_1(s,x_0) - \dot{x}_0\|\,] + Q_3\|x_1(s,x_0) - x_0\| + Q_4\|\dot{x}_1(s,x_0) - \dot{x}_0\|\,]\,ds$$

$$\leq Ή_1 T\ \|x_1(t,x_0) - x_0\| + Ή_2 T\ \|\dot{x}_1(t,x_0) - \dot{x}_0\| \qquad \ldots (18)$$

From (16) and (18) ,the following inequality holds

$$\|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \leq Ή_1 T\ \|x_m(t,x_0) - x_{m-1}(t,x_0)\| + Ή_2 T\ \|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\|\ , \qquad \ldots (19)$$

and from the sequence of function ( 13), when $m = 1$ , we get

$$\|x_2(t,x_0) - x_1(t,x_0)\|$$

$$\leq \int_0^t \int_0^t K[\,\|x_1(s,x_0) - x_0\| + \|\dot{x}_1(s,x_0) - \dot{x}_0\| + Q_1\|x_1(s,x_0) - x_0\| + Q_2\|\dot{x}_1(s,x_0) - \dot{x}_0\| + Ph\,\|x_1(s,x_0) - x_0\| + \|\dot{x}_1(s,x_0) - \dot{x}_0\| + Q_3\|x_1(s,x_0) - x_0\| + Q_4\|\dot{x}_1(s,x_0) - \dot{x}_0\|\,]\,ds\,ds$$

$$\leq \frac{Ή_1 T^2}{2}\ \|x_1(t,x_0) - x_0\| + \frac{Ή_2 T^2}{2}\ \|\dot{x}_1(s,x_0) - \dot{x}_0\| \qquad \ldots (20)$$

Also by(17)and (20) ,the following inequality holds

$$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \leq \frac{Ή_1 T^2}{2}\ \|x_m(t,x_0) - x_{m-1}(t,x_0)\| + \frac{Ή_2 T^2}{2}\ \|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| \qquad \ldots (21)$$

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Now, rewrite the inequalities ( 20) and ( 21) in a vector from:-

$$\begin{pmatrix} \|x_{m+1}(\mathrm{t},x_0) - x_m(\mathrm{t},x_0)\| \\ \|\dot{x}_{m+1}(\mathrm{t},x_0) - \dot{x}_m(\mathrm{t},x_0)\| \end{pmatrix} \le \begin{pmatrix} \frac{'\mathrm{H}_1 T^2}{2} & \frac{'\mathrm{H}_2 T^2}{2} \\ '\mathrm{H}_1 T & '\mathrm{H}_2 T \end{pmatrix} \begin{pmatrix} \|x_m(\mathrm{t},x_0) - x_{m-1}(\mathrm{t},x_0)\| \\ \|\dot{x}_m(\mathrm{t},x_0) - \dot{x}_{m-1}(\mathrm{t},x_0)\| \end{pmatrix}$$

That is

$V_{m+1}(\mathrm{t},x_0) \le \Lambda$ (t) $V_m(\mathrm{t},x_0)$

where $V_{m+1}(\mathrm{t},x_0) = \begin{pmatrix} \|x_{m+1}(\mathrm{t},x_0) - x_m(\mathrm{t},x_0)\| \\ \|\dot{x}_{m+1}(\mathrm{t},x_0) - \dot{x}_m(\mathrm{t},x_0)\| \end{pmatrix}$,

$$\Lambda\,(\mathrm{t}) = \begin{pmatrix} \frac{'\mathrm{H}_1 t^2}{2} & \frac{'\mathrm{H}_2 t^2}{2} \\ '\mathrm{H}_1 t & '\mathrm{H}_2 t \end{pmatrix}$$

and

$$V_m(\mathrm{t},x_0) = \begin{pmatrix} \|x_m(\mathrm{t},x_0) - x_{m-1}(\mathrm{t},x_0)\| \\ \|\dot{x}_m(\mathrm{t},x_0) - \dot{x}_{m-1}(\mathrm{t},x_0)\| \end{pmatrix}$$

If we assuming the max. of $\Lambda$ (t) in [0,T] we have ($\Lambda = \max_{\mathrm{t}\in[0,T]} \Lambda$ (t) )

which leads to the estimate:-

$$\sum_{i=1}^{m} V_i \le \sum_{i=1}^{m} \Lambda^{i-1}\ V_1 \qquad \ldots (22)$$

where $V_1 = \begin{pmatrix} \frac{\mathrm{M}\ T^2}{2} \\ \mathrm{M}\ T \end{pmatrix}$

Since the matrix $\Lambda$ has maximum eigen-values

$\lambda_1 = 0$ and $\lambda_2 = \frac{'\mathrm{H}_1 T^2 + 2'\mathrm{H}_2 T}{2} < 1$

Then the series (22) is uniformly convergent, i. e.

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Lambda^{i-1} V_1 = \sum_{i=1}^{\infty} \Lambda^{i-1} V_1 = (E - \Lambda)^{-1}\, V_1 \qquad \cdots (23)$$

The limiting relation (23) signifies a uniform convergence of the sequences of functions $x_m(\mathrm{t},x_0)$ and $\dot{x}_m(\mathrm{t},x_0)$ in the domain (3.15) as $m \to \infty$ .

Let

$$\left.\begin{aligned}\lim_{m\to\infty} x_m(t,x_0) = x(t,x_0)\\ \lim_{m\to\infty} \dot{x}_m(t,x_0) = \dot{x}(t,x_0)\end{aligned}\right] \quad \cdots (24)$$

Next , we need to prove $x(\mathrm{t}, x_0) \in D$ and $\dot{x}_\infty(\mathrm{t}, x_0) \in D_1$, for all t $\in [0, T]$

Taking

$$\left\| \int_0^t \int_0^t [\, f(s, x_m(s,x_0), \dot{x}_m(s,x_0), Ax_m(s,x_0)\,, A\dot{x}_m(s,x_0), \int_0^{h(s)} g\ (\tau, x_m(s,x_0),\right.$$

$$\dot{x}_m(\tau,x_0), Bx_m(\tau,x_0), B\dot{x}_m(\tau,x_0)\ d\tau\,]\ ds\ ds - \int_0^t \int_0^t [f(s, x(s,x_0), \dot{x}(s,x_0),$$

$$\left. Ax(s,x_0), A\dot{x}(s,x_0), \int_0^{h(s)} g\ \big(\tau, x(\tau,x_0), \dot{x}(\tau,x_0)\ Bx(\tau,x_0), B\dot{x}(\tau,x_0)\big) d\tau\,] ds\ ds \right\|$$

$$\le \int_0^t \int_0^t K[\ \|x_m(s,x_0) - x(s,x_0)\| + \|\dot{x}_m(s,x_0) - \dot{x}(s,x_0)\|$$

$$+ Q_1\|x_m(s,x_0) - x(s,x_0)\| + Q_2\|\dot{x}_m(s,x_0) - \dot{x}(s,x_0)\|$$

$$+\ Ph\ (\|x_m(s,x_0) - x(s,x_0)\| + \|\dot{x}_m(s,x_0) - \dot{x}(s,x_0)\|$$

$$+ Q_3\|x_m(s,x_0) - x(s,x_0)\| + Q_4\|\dot{x}_m(s,x_0) - \dot{x}(s,x_0)\|\ )]\ ds\ ds$$

$$\le \int_0^t \int_0^t [\, K(\,1 + \mathrm{Q}_1 + \mathrm{P\,h} + \ \mathrm{Q}_3\mathrm{P\,h}\,)\|x_m(s,x_0) - x(s,x_0)\|$$

$$K(1 + \mathrm{Q}_2 + \mathrm{Ph} + \mathrm{Q}_4\mathrm{P\,h}\,)\|\dot{x}_m(s,x_0) - \dot{x}(s,x_0)\|\,]\ \ ds\ ds$$

$$\le \acute{\mathrm{H}}_1 \int_0^t \int_0^t \|x_m(s,x_0) - x(s,x_0)\|\ ds\ ds + \acute{\mathrm{H}}_2 \int_0^{\mathrm{t}} \int_0^{\mathrm{t}} \|\dot{x}_m(s,x_0)$$

$$- \dot{x}(s,x_0)\|\ ds\ ds$$

From ( 24), we assume that

$\|x_m((t,x) - \ x(t,x)\ \| \le \in_1$ and $\|\dot{x}_m((t,x) - \ \dot{x}(t,x)\ \| \le \in_1$

There fore

$$\left\| \int_0^t \int_0^t [\, f(s, x_m(s,x_0), \dot{x}_m(s,x_0), Ax_m(s,x_0)\,, A\dot{x}_m(s,x_0), \int_0^{h(s)} g\ (\tau, x_m(s,x_0),\right.$$

$$\dot{x}_m(\tau, x_0), Bx_m(\tau, x_0), B\dot{x}_m(\tau, x_0)\, d\tau\, ]\, ds\, ds - \int_0^t \int_0^t [f(s, x(s, x_0), \dot{x}(s, x_0),$$

$$Ax(s, x_0), A\dot{x}(s, x_0), \int_0^{h(s)} g\left(\tau, x(\tau, x_0), \dot{x}(\tau, x_0)\, Bx(\tau, x_0), B\dot{x}(\tau, x_0)\right) d\tau\, ] ds\, ds \Big\|$$

$$\leq \acute{H}_1 \int_0^t \int_0^t (\in_1)\, ds\, ds + \acute{H}_2 \int_0^t \int_0^t (\in_1) ds\, ds$$

$$\leq \in_1 \left( \frac{(\acute{H}_1 + \acute{H}_2)\, T^2}{2} \right)$$

$\leq \in$ , for all $m \geq 0$ , where $\in_1 = \frac{2\in}{(\acute{H}_1 + \acute{H}_2)\, T^2}$

So that

$$\lim_{m\to\infty} \int_0^t \int_0^t [\, f(s, x_m(s, x_0), \dot{x}_m(s, x_0), Ax_m(s, x_0)\,, A\dot{x}_m(s, x_0),$$

$$\int_0^{h(s)} g\,(\tau, x_m(\tau, x_0), \dot{x}_m(\tau, x_0)\,, Bx_m(\tau, x_0), B\dot{x}_m(\tau, x_0)\, d\tau\, ]\, ds\, ds$$

$$- \int_0^t \int_0^t [\, f(s, x(s, x_0), \dot{x}(s, x_0), Ax(s, x_0)\,, A\dot{x}(s, x_0),$$

$$\int_0^{h(s)} g\,(\tau, x(\tau, x_0), \dot{x}(\tau, x_0)\,, Bx(\tau, x_0), B\dot{x}(\tau, x_0)\, d\tau\, ]\, ds\, ds$$

$$= \int_0^t \int_0^t [\, f(s, x_m(s, x_0), \dot{x}_m(s, x_0), Ax_m(s, x_0)\,, A\dot{x}_m(s, x_0),$$

$$\int_0^{h(s)} g\,(\tau, x_m(\tau, x_0), \dot{x}_m(\tau, x_0)\,, Bx_m(\tau, x_0), B\dot{x}_m(\tau, x_0)\, d\tau\, ]\, ds\, ds$$

$$- \int_0^t \int_0^t [\, f(s, x(s, x_0), \dot{x}(s, x_0), Ax(s, x_0)\,, A\dot{x}(s, x_0),$$

$$\int_0^{h(s)} g\,(\tau, x(\tau, x_0), \dot{x}(\tau, x_0)\,, Bx(\tau, x_0), B\dot{x}(\tau, x_0)\, d\tau\, ]\, ds\, ds$$

Thus $x(\text{t}, x_0) \in D$ , $\dot{x}_\infty(\text{t}, x_0) \in D_1$ and $x(\text{t}, x_0)$ is a solution of (P).

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**Theorem 3.(Uniqueness Theorem ).**

If the solution of the problem (P). satisfying the inequalities and all conditions of theorem 2 , then the solution $x(\mathrm{t}, x_0)$ is a unique of the problem (P).

**Proof.** Suppose that $r(t, x_0)$ is another conditions solution of (P), then

$$r(\mathrm{t}, x_0) = x_0 + \dot{x}_0 t + \int_0^t \int_0^t [f(s, r(s, x_0), \dot{r}(s, x_0), Ar(s, x_0), A\dot{r}(s, x_0),$$
$$\int_0^{h(s)} g\,(\tau, r(\tau, x_0), \dot{r}(\tau, x_0), Br(\tau, x_0), B\dot{r}(\tau, x_0) d\tau\,]\, ds\, ds$$

and

$$\dot{r}(\mathrm{t}, x_0) = \dot{x}_0 + \int_0^t [f(s, r(s, x_0), \dot{r}(s, x_0), Ar(s, x_0), A\dot{r}(s, x_0),$$
$$\int_0^{h(s)} g\,(\tau, r(\tau, x_0), \dot{r}(\tau, x_0), Br(\tau, x_0), B\dot{r}(\tau, x_0) d\tau\,]\, ds$$

Taking

$$\|x(t, x_0) - r(\mathrm{t}, x_0)\|$$
$$\leq \int_0^t \int_0^t [\,K(\,1 + \mathrm{Q}_1 + \mathrm{P\,h} + \mathrm{Q}_3 \mathrm{P\,h}\,)\|x(\mathrm{t}, x_0) - r(t, x_0)\,\|$$
$$+ K(1 + \mathrm{Q}_2 + \mathrm{P\,h} + \mathrm{Q}_4 \mathrm{P\,h}\,)\,\|\dot{x}(t, x_0) - \dot{r}(t, x_0)\,\|]\, ds\, ds$$
$$\leq \frac{Ή_1 T^2}{2} \|x(\mathrm{t}, x_0) - r(t, x_0)\,\| + \frac{Ή_2 T^2}{2} \|\dot{x}(t, x_0) - \dot{r}(t, x_0)\,\| \qquad \ldots$$

(25)

and

$$\|\dot{x}(t, x_0) - \dot{r}(t, x_0)\,\| = \Bigg\| x_0 + \int_0^t [f(s, x(s, x_0), \dot{x}(s, x_0), Ax(s, x_0), A\dot{x}(s, x_0),$$
$$\int_0^{h(s)} g\,(\tau, x(\tau, x_0), \dot{x}(\tau, x_0)\, Bx(\tau, x_0), B\dot{x}(\tau, x_0)\,)\, d\tau]\, ds - x_0 - \int_0^t [f(s, r(s, x_0),$$

$$\dot{r}(s,x_0), Ar(s,x_0), A\dot{r}(s,x_0), \int_0^{h(s)} g\ (\tau, r(\tau,x_0), \dot{r}(\tau,x_0)\ Br(\tau,x_0), B\dot{r}(\tau,x_0))d\tau]\ ds\|$$

$$\leq \int_0^t \Big\| f(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0), A\dot{x}(s,x_0), \int_0^{h(s)} g\ (\tau, x(\tau,x_0), \dot{x}(\tau,x_0),$$

$$Bx(\tau,x_0), B\dot{x}(\tau,x_0))\ d\tau - f(s, r(s,x_0), \dot{r}(s,x_0), Ar(s,x_0), A\dot{r}(s,x_0),$$

$$\int_0^{h(s)} g\ (\tau, r(\tau,x_0), \dot{r}(\tau,x_0)\ Br(\tau,x_0), B\dot{r}(\tau,x_0)\ d\tau \Big\|\ ds$$

$$\leq 'H_1 T\ \|x(t,x_0) - r(t,x_0)\| + 'H_2 T\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \qquad \ldots$$

(26)

From (25) and (26) we have

$$\begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix} \leq \begin{pmatrix} \frac{'H_1 T^2}{2} & \frac{'H_2 T^2}{2} \\ 'H_1 T & 'H_2 T \end{pmatrix} \begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix}$$

By the condition $\lambda_{max}(\Lambda) < 1$ , then

$$\begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix} < \begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix},$$

This is contradiction , then $\begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Therefore $x(t,x_0) = r(t,x_0)$ , $\dot{x}(t,x_0) = \dot{r}(t,x_0)$ and hence $x(t,x_0)$ is a unique solution of the problem (P) .

**III.Stability solution of (P).**

In this section, we study the stability theorem for the solution of the problem (P).

**Theorem 4. (stability theorem)** : Assume that all inequalities and conditions of theorem 1 are satisfies, then the solution of the problem( P), is stable for all $t \geq 0$ .

**Proof.** Let $r(t,x_0)$ and $\dot{r}(t,x_0)$ be any two solutions of the problem (P).

Then

$$z(\mathrm{t},x_0) = z_0 + \dot{z}_0 t + \int_0^t \int_0^t [f(s,z(s,x_0),\dot{z}(s,x_0),Az(s,x_0)\,,A\dot{z}(s,x_0),$$
$$\int_0^{h(s)} g\ (\tau,z(\tau,x_0),\dot{z}(\tau,x_0)\,,Bz(\tau,x_0),B\dot{z}(\tau,x_0)d\tau\,]\,ds\,ds$$

and

$$\dot{z}(\mathrm{t},x_0) = \dot{z}_0 + \int_0^t [f(s,z(s,x_0),\dot{z}(s,x_0),Az(s,x_0),A\dot{z}(s,x_0),$$
$$\int_0^{h(s)} g\ (\tau,z(\tau,x_0),\dot{z}(\tau,x_0)\,,Bz(\tau,x_0),B\dot{z}(\tau,x_0)d\tau\,]\,ds$$

Taking

$$\|x(s,x_0) - z(\mathrm{t},x_0)\| = \Big\| x_0 + \dot{x}_0 t + \int_0^t \int_0^t [\,f(t,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),$$
$$A\dot{x}(s,x_0),\int_0^{h(t)} g\ (s,x(s,x_0),\dot{x}(s,x_0),Bx(s,x_0),B\dot{x}(s,x_0)\ ds\,]ds\,ds - z_0 - \dot{z}_0 t$$
$$-\int_0^t \int_0^t [\,f(t,z(s,x_0),\dot{z}(s,x_0),Az(s,x_0)\,,A\dot{z}(s,x_0),\int_0^{h(t)} g\ (s,z(s,x_0),$$
$$\dot{z}(s,x_0)\,,Bz(s,x_0),B\dot{z}(s,x_0)\ ds\,]\,ds\,ds\|$$

$$\leq \|x_0 + \dot{x}_0 t - z_0 - \dot{z}_0 t\| + \int_0^t \int_0^t \|f(t,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),$$
$$\int_0^{h(t)} g\ (s,x(s,x_0),\dot{x}(s,x_0)\ Bx(s,x_0),B\dot{x}(s,x_0)\ ds - f(t,z(s,x_0),\dot{z}(s,x_0),$$
$$Az(s,x_0),A\dot{z}(s,x_0),\int_0^{h(t)} g\ (s,z(s,x_0),\dot{z}(s,x_0)\ Bz(s,x_0),B\dot{z}(s,x_0)\ ds\Big\|\ ds\,ds$$

By the definition of stability for $\|x_0 + \dot{x}_0 t - z_0 - \dot{z}_0 t\| \leq \delta_1$ we get

$$\leq \delta_1 + \frac{Ή_1 T^2}{2}\|x(\mathrm{t},x_0) - z(t,x_0)\| + \frac{Ή_2 T^2}{2}\|\dot{x}(t,x_0) - \dot{z}(t,x_0)\|, \qquad \dots (27)$$

$$\|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| = \Big\| x_0 + \int_0^t [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),$$

$$\int_0^{h(s)} g\,(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,Bx(\tau,x_0),B\dot{x}(\tau,x_0)\,)\,d\tau]\,ds - z_0 - \int_0^t [f(s,z(s,x_0),$$

$$\dot{z}(s,x_0),Az(s,x_0),A\dot{z}(s,x_0),\int_0^{h(s)} g\,(\tau,z(\tau,x_0),\dot{z}(\tau,x_0)\,Bz(\tau,x_0),B\dot{z}(\tau,x_0))d\tau]\,ds\|$$

$$\leq \|x_0 - z_0\| + \int_0^t \|f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),$$

$$,\int_0^{h(s)} g\,(\tau,x(\tau,x_0),\dot{x}(\tau,x_0),\quad Bx(\tau,x_0),B\dot{x}(\tau,x_0))\,d\tau - f(s,z(s,x_0),\dot{z}(s,x_0)$$

$$,Az(s,x_0),A\dot{z}(s,x_0),\int_0^{h(s)} g\,(\tau,z(\tau,x_0),\dot{z}(\tau,x_0)\,Bz(\tau,x_0),B\dot{z}(\tau,x_0)\,d\tau\Big\|\,ds$$

And also by the definition of stability for $\|x_0 - z_0\| \leq \delta_2$ we get

$$\leq \delta_2 + 'H_1 T\,\|x(\mathrm{t},x_0) - r(t,x_0)\| + 'H_2 T\,\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \qquad \dots (28)$$

From (27) and (28) we find that

$$\begin{pmatrix} \|x(\mathrm{t},x_0) - z(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| \end{pmatrix}$$

$$\leq \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \frac{'H_1 T^2}{2} & \frac{'H_2 T^2}{2} \\ 'H_1 T & 'H_2 T \end{pmatrix} \begin{pmatrix} \|x(\mathrm{t},x_0) - z(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| \end{pmatrix}$$

And hence

$$\begin{pmatrix} \|x(\mathrm{t},x_0) - z(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| \end{pmatrix} \leq \begin{pmatrix} \in_1 \\ \in_2 \end{pmatrix}$$

So that the solution of (P) is stable for all $\mathrm{t} \in [0,T]$

**V.Existence and uniqueness solution of (P) .**

In this section, we prove the existence and uniqueness theorem of (P) by using Banach fixed point theorem .

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**Theorem 5. (Banach fixed point theorem)**. Let the vector functions $f(t,x,\dot{x},y,\dot{y},z)$ and $g(t,x,\dot{x},w,\dot{w})$ of the problem (P) are defined and continuous on the domain (1) and satisfies assumptions and all conditions of theorem 1 . Then the problem (P) has a unique continuous solution on the domain (1).

**Proof** .Let (C [0,T] , ‖.‖ ) be a Banach space and $T^*$ be a mapping on C[0,T] as follows :-

$$T^*x(t,x_0) = x_0 + \dot{x}_0 t + \int_0^t \int_0^t [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0)\,,A\dot{x}(s,x_0),$$

$$\int_0^{h(s)} g\,(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0)d\tau\,]\,ds\,ds$$

and

$$T^*\dot{x}(t,x_0) = \dot{x}_0 + \int_0^t [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0)\,,A\dot{x}(s,x_0),$$

$$\int_0^{\dot{h}(s)} g\,(\tau,x(\tau,x_0),x(\tau,x_0),Bx(\tau,x_0),B\dot{x}(\tau,x_0)d\tau\,]\,ds$$

Since $\int_0^{h(s)} g\,(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0)\,)d\tau$ is continuous on the same domain (1) and also

$$\int_0^t \int_0^t [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0)\,,A\dot{x}(s,x_0),$$

$$\int_0^{h(s)} g\,(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0)d\tau\,]\,ds\,ds$$

is continuous on the same domain .

There fore $T^*$: C [0,T] $\longrightarrow$ C [0,T] .

Now, we shall to prove that $T^*$ is a contraction mapping on C [0,T] .

Let $x(\mathrm{t}, x_0)$ , $z(\mathrm{t}, x_0)$ be a vector functions on C [0,T] , then

$$\| T^* x(\mathrm{t}, x_0) - T^* z(\mathrm{t}, x_0) \|$$

$$\leq \max_{t\in[0,T]} \int_0^t \int_0^t | f(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0), A\dot{x}(s,x_0), \int_0^{h(s)} g\ (\tau, x(\tau,x_0), \dot{x}(\tau,x_0), Bx(\tau,x_0), B\dot{x}(\tau,x_0)\ )\ d\tau - f(s, z(s,x_0), \dot{z}(s,x_0), Az(s,x_0), A\dot{z}(s,x_0), \int_0^{h(s)} g\ (\tau, z(\tau,x_0), \dot{z}(\tau,x_0)\ , Bz(\tau,x_0), B\dot{z}(\tau,x_0))\ d\tau \mid ds\ ds$$

$$\leq \frac{'\mathrm{H}_1 T^2}{2} \|x(\mathrm{t}, x_0) - z(t,x_0)\| + \frac{'\mathrm{H}_2 T^2}{2} \|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| \quad \ldots$$

(29)

and also by the same method ,we have

$$\| T^* \dot{x}(t,x_0) - T^* \dot{z}(t,x_0) \|$$

$$\leq \max_{t\in[0,T]} \int_0^t |f(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0), A\dot{x}(s,x_0), \int_0^{h(s)} g\ (\tau, x(\tau,x_0), \dot{x}(\tau,x_0), Bx(\tau,x_0), B\dot{x}(\tau,x_0)\ )d\tau - f(s, z(s,x_0), \dot{z}(s,x_0), Az(s,x_0), A\dot{z}(s,x_0), \int_0^{h(s)} g\ (\tau, z(\tau,x_0), \dot{z}(\tau,x_0)\ , Bz(\tau,x_0), B\dot{z}(\tau,x_0))d\tau \mid ds$$

$$\leq {'\mathrm{H}_1} T \|x(\mathrm{t}, x_0) - z(t,x_0)\| + {'\mathrm{H}_2} T\ \|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| \quad \ldots (30)$$

Rewrite ( 29) and (30) in a vector form

$$\begin{pmatrix} \|T^* x(\mathrm{t}, x_0) - T^* z(t,x_0)\| \\ \|T^* \dot{x}(t,x_0) - T^* \dot{z}(t,x_0)\| \end{pmatrix} \leq \begin{pmatrix} \frac{'\mathrm{H}_1 T^2}{2} & \frac{'\mathrm{H}_2 T^2}{2} \\ '\mathrm{H}_1 T & '\mathrm{H}_2 T \end{pmatrix} \begin{pmatrix} \|x(\mathrm{t}, x_0) - z(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| \end{pmatrix}$$

By the condition $\lambda_{max}\,\Lambda < 1$ , then $T^*$ is a contraction mapping ,thus by Banach fixed point theorem, then there exists a fixed point $x(\mathrm{t}, x_0)$ in C[0,T] , such that $T^* x(\mathrm{t}, x_0) = x(\mathrm{t}, x_0)$ .

There fore

$$x(t,x_0) = x_0 + \dot{x}_0 t + \int_0^t \int_0^t [f(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0), A\dot{x}(s,x_0),$$

$$\int_0^{h(s)} g\,(\tau, x(\tau,x_0), \dot{x}(\tau,x_0), Bx(\tau,x_0), B\dot{x}(\tau,x_0)d\tau\,]\,ds\,ds$$

is a unique continuous solution of the problem (P) .

# References

[1].Apostal, Tom M., "Mathematical Analysis", 2nd edition Addision-Wesley publishing Company, Inc, London, (1964).
[2]. Butris, R. N. and Rafeq, A. Sh., Existence and Uniqueness Solution for Non-linear Volterra Integral Equation, J. Duhok Univ. Vol. 14, No. 1, (Pure and Eng. Sciences),(2011).
[3].Butris, R. N. and Dawoud S.Abdullh., Solution of Integro-differential Equation of the first order with the operators, International Journal of Scientific Engineering and Applied Science ,India,(IJSEAS) , Volume 1, lssue6,(2015).
[4] Coddington, E. A., and Levinson, N., Theory of ordinary differential Equations, Mc Graw- Hill Book Company , New York, (1995).
[5] Rama, M. M., Ordinary Differential Equations Theory and Applications, Britain, (1981).
[6] Royden, H. L., Real Analysis, Prentice-Hall of India Private Limited, New Delhi, (2005) .
[7].Struble, R. A., "Nonlinear differential equations", McGraw-Hill Book Company, Inc, New York, (1962).
[8].Moulay Rchid Sidi Ammi, El Hassan El Kinani, Delfim F. M. Torres, "Existence and uniqueness of solution to a functional integro-differential fractional equation" Electronic Journal of Differential Equations, (2012).