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How do First-Year-University Students use their Reasoning and Intuition to solve Geometry Problems?

Luis Weng San

Department of Mathematics and Informatics, Eduardo Mondlane University
Maputo – Mozambique

Abstract

The present study was designed to explore students' algebraic and geometric knowledge and thinking and their ability to access and use algebraic knowledge and thinking in geometry problem solving. For this purpose, it was constructed a 50-minute written test and administered to 26 first-year students at Pedagogic University (UP) in October 2014 (UP mainly aims at training secondary and high school pre-service teachers for different subjects including mathematics). After sorting them according to their background, 7 students were then selected for deeper analysis of their written responses. For analysis of students' written responses seven test scripts were sampled based on their different academic, professional backgrounds their responses.

Keywords**:** *algebraic and geometric knowledge, reasoning, intuition, problem solving.*

1. Introduction

It has been recognized by mathematics education researchers that there are no definite answers to questions about what constitutes "mathematical understanding" [17]. However, one can only notice some "manifestations" of this complex phenomenon. Thus, [23] characterizes such manifestations as models of understandings. In order for an individual to be acquainted with the models of understanding, and therefore, to (partially) understand what mathematical understanding is, it seems important to refer to what the individual intends to understand – 'the object of understanding', mathematics in this case. For instance, according to [6], all mathematical facts are proved by deduction from some initial sets of assumptions, or *axioms* (from the Latin *axioma*, meaning 'a principle'). The axioms are assumed to be, or at least widely accepted within the mathematical community, obvious truths. In other words, all mathematical facts are constructed from the basic ones by proof. Thus, mathematics can metaphorically be considered as a 'building'. It is from this perspective that understanding in mathematics can be portrayed as cumulative structuring [17].

That is, mathematics is a networked framework of ideas, where understanding of simpler ideas (axioms) forms a foundation or anchor point for higher-level ideas (propositions and theorems). Also, [6] stated that the axioms are like the foundations of a building. In common language, one can portray mathematical understanding as cumulative structuring by recognizing the need to know how the 'building' of mathematics is constructed. [6], quoting Galileo, contends: "The great book of nature can be read only by those who know the language in which it is written. And this language is mathematics" (p.10). Mathematics is also considered as a 'language'. And language is a social tool. Hence, in order for one to know the 'language' of mathematics, in other words, to understand mathematics, one must be involved in social processes through discursive practice in particular settings.

Thus, mathematical understanding may be portrayed as a social process [17]. Within this context, one can develop fluency of the 'language' of mathematics at different levels depending on the different forms of meaning held and developed simultaneously but individually. In this way, understanding may be seen as a form of knowing [17].

Meanwhile, there are claims that Euclidean geometry might itself be a stumbling block for students' understanding and performance in geometry in general. [15:308] supports this claim when he asserts that: *"Descartes was disturbed by the fact that every proof in Euclidean geometry called for some new, often ingenious, approach. He explicitly criticized the geometry of the ancients as being too abstract, and so much tied to figures"*. Moreover Descartes criticized synthetic geometry (the geometrical arguments of Euclid and Apollonius) for lacking a general method (strategy). According to [13] Descartes achieved this goal of finding a general method in geometry by the introduction of coordinate systems and the creation of analytic geometry (coordinate geometry).

On one hand, [26] stated that iconic processing (visual and spatial imagery) which served students well at an earlier stage might later become an impediment for the development of the formal theory in the student's mind. And, [20] found that the use of iconic support in mathematical problem solving is of paramount importance, wherever students meet a certain concept for the first time. The finding of [20] supports the idea that there is a need to move along the continuum to abstraction so that image schemata – a bridge between abstract logical structures and particular concrete images and experience – may become more flexible and abstract. Similar perspectives can be found in [28] and [3]. On the other hand, [15] stated that, for example, in synthetic geometry, to prove that the altitudes of a triangle meet in a point, intersections inside and outside the triangle are considered separately. In analytic geometry they are considered together.

[7] also asserts that "many [geometric] problems are easier to solve using coordinate geometry (analytic geometry), transformations or vectors than using traditional Euclidean geometry" (p. 6). An example to illustrate this idea is as follows. If we are to show that a line which bisects two sides of a triangle is parallel to the third side in the Euclidean geometry, then we can use, for example, the following knowledge schemas: We need to use an auxiliary theorem which states that a line bisecting one side of a triangle and is parallel to a second side bisects the third side of the triangle. Afterwards, it is recommendable to approach the theorem using an indirect proof, where at each step what we say has a meaning in relation to a figure [16:325, 326]. While in the coordinate geometry context, we can use a direct proof, approaching it, for example, by setting any coordinates for the vertexes of the triangle and using collinear vectors and formulae of midpoint coordinates. It might be simpler to use vector algebra geometrically. Analyzing the solutions we can conclude that using traditional Euclidean geometry requires one to be acquainted with previously demonstrated theorems, postulates and axioms. Besides, one needs to make complicated connections and be bound to figures at each step. However, using coordinate geometry requires one to know vector algebra equations, which are simpler to deduce and to recall. Afterwards, one applies them to directly solve the problem. In other words through coordinate geometry we can solve the geometric problems in more elegant, quick, and fuller way than in Euclidean geometry.

On geometric understanding, [25:105] described geometry as a dual subject in its essence, "First there is the visual, self-contained, synthetic side, which seems intuitively natural; then the algebraic, analytic side, which takes over when intuition fails and integrates geometry into the larger world of mathematics". [25:37] also described geometric intuition as "our imagination (that) leads us to conclusions via steps that 'look right' but may not have a purely logical basis". Meanwhile, he asserted that the results seen by intuition should be validated by logic. One of the attempts to validate intuition results is the so-called synthetic geometry. Although in this system, all theorems are derived by pure logic from a list of visually plausible axioms, those theorems are close to intuition that is the step in a proof imitates the way we see a theorem. In turn [12] characterized intuition as a type of cognition that seeks self-evidence, immediacy, and certitude. She added saying that models are a central factor of intuition in mathematics. Fischbein (1987) as quoted by [12] called them intuitive models. She considered geometric intuition the use of models stemming from geometry. Because of the nature of geometry, she stated that a geometric model is always associated with a figural model. In her study the figural model she used corresponded to the use of drawings that mean pictorial representations. Similarly, in my study I associate geometric intuition with visualization and construction of (mental or physical) pictures. [10:23] corroborated the relation between the synthetic aspect of geometry and intuition stating that "Intuition… means a global, synthetic grasp and interpretation of a situation".

Thus the dual nature of geometry makes the subject extremely rich. It enables the interplay and interconnection between mathematical language (e.g. algebra) and the language of pictures, between the synthetic approach (where at each step what you say has a meaning in relation to a figure) and the analytic approach (using coordinates to facilitate transfer to a numeric or algebraic framework, which allows blind calculation) [8: 26, 27]. According to [8], a critical aspect necessary for the development of a more complete understanding [of a geometric concept] is to understand the equivalence between all its definitions, to have them all accessed and made available at each moment, to be able to conveniently choose one or the other, and to transfer properties from one framework to another. This is an important proficiency particularly for students studying geometry.

For this proficiency in geometry to be realized, an individual needs to develop three kinds of cognitive processes that fulfill specific epistemological functions. A framework that [9] presented recognizes that these cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry. These processes are:

- Visualization processes, for example the visual representation of a geometrical statement, or the

heuristic exploration of a complex geometrical situation;
- Construction processes (using tools); and
- Reasoning in relationship to discursive processes for extension of knowledge, for proof, for explanation (p.38).

Though [9] added up saying these processes can be performed separately and independently. However, these three kinds of cognitive processes are closely connected. And, [14] suggested that the visual reasoning (visualization processes) is much more than an intuitive support for higher level reasoning: it is the backbone of a rigorous proof. She explained (quoting Hanna, 1990) that the visual process includes: 1) a new way of looking at the situation in order to suggest a generalization, 2) its proof and verification in one process, and 3) an explanation of "why" the generalization holds. However, [9] claimed that visual reasoning in some cases can be misleading or impossible [due to its subjectivity]. For example a circle can be seen as an ellipse in another plane than the frontal one. Besides, quadrature of the circle is impossible within ruler-and-compass context (proved by the German mathematician Ferdinand Lindemann in 1882 as quoted by [6]; consequently, within this context visual reasoning is impossible.

Construction processes depend only on connections between mathematical properties and the technical constrains of the tools used [9]. Reasoning takes place when by experimentation [construction by ruler and compass or geometrical softwares] and inductive generalization [by visualization processes], one extends her geometrical knowledge about shapes and relations and extends her "vocabulary" of legitimate ways of reasoning. Deductive reasoning [dependent exclusively on the corpus of propositions – definitions, axioms, and theorems] then becomes a vehicle for understanding and explaining why an inductively discovered conjecture might hold [14].

Another framework describing the development of geometrical thinking that has been the subject of considerable research is the van Hiele model of thinking in geometry (see, for instance, [21]). [21] stated that the foundations of the van Hiele model are in pedagogical "experience" and teaching experiments, and that the justifications offered for it are loose by any rigorous standard and justified theory. Similarly lacking is a detailed explanation of cognitive processes that underlie competent performance in geometry. However, he recognized that this model provides an important starting point for conceptualizing learner growth and understanding in geometry: formal symbolic manipulations in the deductive universe are meaningless for those students who lack a deep intuitive understanding of the properties of the elements in the empirical reference world from which the abstract objects manipulated in the deductive universe are abstracted. This reinforces what was previously mentioned that Euclidean geometry might serve as an introductory subject to geometry to foster a deep intuitive understanding of the properties of the geometric objects. Meanwhile, there might be a need to move along to formal symbolic manipulations in the deductive universe (e. g. analytic geometry) for the development of abstract logical structures (abstract image schemata) in students' mind that may facilitate the development of the formal theory of geometry.

On one hand, [11] added arguing that "it is misleading to try to place a pupil at one (van Hiele) level (globally), since they can be at different levels with different pieces of work. Moreover, the van Hiele levels seem better applied not to the pupil, but to the path needed for teaching a piece of work" (p. 202). Likewise, [22] found that students did not think at the same van Hiele level in all areas of geometry content. Accordingly she advised "research that attempts to sort out the overlap between skill or knowledge and the thinking processes that characterize van Hiele levels should use instruments that are content specific" (p. 320). On the hand, [7] also recognized that the van Hiele model provides a valuable framework for studying geometric thinking. However, this framework mostly targets Euclidean geometry. That is why studies that have used the van Hiele framework have tended to focus on geometry in a purely Euclidean context at the school level. [7] observed that "algebra and algebraic thinking, which has significantly influenced the secondary school geometry curriculum, has not been considered in this [van Hiele] framework" (p. 10).

In general, the main focus of research on the van Hiele theory has been directed at the notion of a hierarchy of five levels of thinking. As modified by Hoffer (1981) quoted by [2] the five van Hiele levels are as follows:

Level 1 (*Visualization*): The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

Level 2 (*Analysis*): The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established. In other words, figures are identified by their mathematical properties. The properties, however, are seen to be independent of one another. They are discovered and generalized from the observation (visualization) of a few examples.

Level 3 (*Abstraction*): The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Level 4 (*Deduction*): The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.

Level 5 (*Rigor*): The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.

As [19] asserted, some researchers such as [2] and Gutiérrez *et al* (1991), had even challenged what van Hiele refers to as one of the most distinctive features of the levels, namely, their discontinuity. As a result two consequences of this theory were observed, namely: *i) Learners may be on different van Hiele levels for different concepts, and ii) the van Hiele levels are not particularly suited for fine-grained analyses of a learner's understanding*. (p. 280)

From the previous descriptions, links can be established between the van Hiele Theory and the conceptual model adapted in this study (Table 1) so as some contributions may be proposed to face some of the shortcomings mentioned earlier.

Table 1: Links between van Hiele levels and the conceptual model adapted in the study

Conceptual Model Adapted	***van Hiele Levels***
Synthetic aspect of geometry	Levels 1 and 2
Analytic aspect of geometry	Levels 2, 3, 4, and 5

According to [25] the synthetic side of geometry is concerned with visualization processes driven by intuition, thus corresponding to van Hiele Level 1 and partly Level 2. Level 2 (partly) to Level 5 seem to characterize the analytic side of geometry as analysis, abstraction, deduction, and rigor are mainly analytic concepts and especially in this case algebraic stances where reasoning plays a key role.

Hence, the concern of this study is not to place the learner's understanding according to the van Hiele levels discontinuously. On the contrary the concern is in placing the learner's understanding in terms of the two aspects of geometry, synthetic and analytic being algebra (algebraic thinking) the bridge between the two aspects in the Mozambique context particularly at Pedagogic University. The research in this study is therefore theoretically significant as it explores how algebra may serve as a tool into geometrical understanding, a conceptual relationship that has not received any focused attention in the conceptualization of the van Hiele model of understanding.

Thus, research on students' thinking in geometry has mostly used the van Hiele's framework [7]. [7] adds saying that this framework targets students' thinking in a purely Euclidean context and ignores the substantial interaction of geometry with algebra. However, my study investigated how first-year university students bring their knowledge and thinking of algebra in understanding and working with geometry. Besides this study dealt with geometry through use of coordinate geometry, transformations, and vectors, which involve algebra. Hence, this study appears to be significant as it used another framework in geometry thinking adapted from the work of [4], [9], [24], and [25]).

Certainly, this framework crosses somehow with van Hiele's, meanwhile it has proper features. Hence, it is expected that the outcomes of this study will provide a basis for a more comprehensive framework for studying students' thinking in geometry and will also aid in the planning and implementation of instruction in geometry at the high school level and especially at Pedagogic University in Maputo – Mozambique.

Accordingly, in this study it was considered the strategies used in the Euclidean Geometry problem solving as were mostly driven by intuition because the theorems are close to intuition. Therefore, the study aimed to explore how students solve geometry problems and to what extent do they use algebraic knowledge and thinking in solving such problems. That is, "*How do First-Year-University Students use their Reasoning (Analytic Strategies) and Intuition (Synthetic Strategies) to solve Geometry Problems?*"

2. Material and Methods

In articulating a perspective for this study, I focus on three key concepts: "thinking", "understanding" and "knowledge connectedness".

The concepts "thinking" and "understanding" have been used interchangeably in the literature relating to mathematical activity. I acknowledge the difficulty of defining fully the terms "thinking" and "understanding".

Moreover, it is not the intention of this study to deal with the explicit defining of terms extensively. However, this study is concerned with some indicators of thinking and understanding. The term “thinking” is conceived as the ability to exercise the mind in order to make a decision. Thus, the term ‘understanding’ is conceived as the ability to know or comprehend the nature or the meaning of something ([5] and [27]). These skills are intertwined and both take place in one’s mind. This implies that in order for an individual to come to know the nature or the meaning of something, she should first exercise the mind to make a decision about that something and vice-versa.

Quoting Burton (1984), [7] defined thinking as the means used by humans to improve their understanding of, and exert some control over their environment. Making meaning in mathematics involves making connections among different concepts in mathematics, that is “… the forging of connections across domains [in mathematics]” ([18:130]).

Students’ ability to make meaning and establish connections – “bridges” – between concepts is dependent on the nature of the subject matter they are working with and the levels of thinking and understanding that is being demanded in the problem-solving context related to the subject.

The present study was designed to explore students’ algebraic and geometric knowledge and thinking and their ability to access and use algebraic knowledge and thinking in geometry problem solving. For this purpose, it was constructed a 50-minute written test and administered to 26 first-year students at Pedagogic University (UP) in October 2014 (UP mainly aims at training secondary and high school pre-service teachers for different subjects including mathematics).

After sorting them according to their background, 7 students were then selected for deeper analysis of their written responses. The tasks of the test were developed by the researcher with adaptations from different sources ([16] and [1]) and validated by the lecturer of Euclidean and Analytic Geometry courses at UP.

For analysis of students’ written responses seven test scripts were sampled based on their different academic, professional backgrounds and the richness of their responses.

Thus, as the subjects for the study consisted of 26 first-year university students, were also sorted according to their background and it was found that these subjects came from different academic and professional (pedagogic) backgrounds as shown in the following Table 2.

So, according to the differences in terms of teaching and learning experiences and contexts it was also expected differences in the approaches the subjects used in the problem solving situations particularly in geometric tasks.

Table 2: Academic and professional background of subjects

Number of students/Category	***Academic background***	***Professional background***
10 / (I)	***Grade 12***	***None***
1 / (II)	Grade 12	Grade 7+ 2 years lower primary school teacher training course
2 / (III)	Grade 12	Grade 10+2 years upper primary school teacher training course
1 / (IV)	Grade 12	Grade 10+3 years technical college teacher training
8 / (V)	Grade 10	Grade 10+2 years upper primary school teacher training course
4 / (VI)	Grade 10	Grade 10+3 years technical college teacher training

3. The data collection instrument

The written test used consisted of two tasks as presented below:

In Portuguese	***<u>Exercício 1</u>****: - Seja dado um triângulo [ABC], onde o segmento DE bisecta os lados AB e AC.* *a) Construa o triângulo [ABC] e o segmento DE.* *b) Conjectura a(s) relação(ões) entre os segmentos DE e BC.* *c) Demonstre essa(s) relação(ões).*
In English	***<u>Task 1</u>****: - In triangle [ABC], DE bisects AB and AC.* *a) Sketch [ABC] and DE.* *b) What relation(s) can you write connecting DE and BC?* *c) Prove the relation(s).]*

In Portuguese	***Exercício 2***: *- Seja dado um polígono com n vértices.* a) *Deduza a formula do número total de diagonais do polígono.* b) *Calcule o número total de diagonais dum polígono de 70 vértices*
English translantion	***Task 2***: *- A polygon possesses n vertexes.* a) *Write a formula, which indicates the total number of its diagonals.* b) *Determine the number of diagonals of 70- vertex polygon*.]

Task 1 aimed at assessing the underlying cognitive processes in geometry (visualization, construction, and reasoning processes) so as in algebra (symbolization, relations, modeling and generalizations). The students were expected to construct a triangle under given condition; to visualize some relations between segment lines DE and BC; and finally to reason how to prove those relations found. For proving it was expected the students to use either vector algebra (Analytic Geometry) or Euclidean Geometry knowledge as they might have treated both topics either at school or at the university according to the syllabuses.

On the one hand in Euclidean Geometry domain the students might have used the theorem on bisection and parallelism as an auxiliary theorem and using indirect proof (see the solution of this task in [16]). In this solution symbolization of segments and the relations of their lengths play a fundamental role. While, on the other hand in Analytic Geometry domain the students might have added vectors and known the concept of co-linearity of vectors and interpreted it as the parallelism of the respective segments. After proving the students were expected to generalize the result to any other triangle under the same conditions. This aim might be assessed when they encounter a task where they should use this theorem.

As we can see these concepts (symbolization of segments, relations of the segment lengths, vector addition, and co-linearity of vectors) evoke algebraic thinking (symbolization, relations, modeling and generalization). The purpose of Task 2 was to assess how students construct and visualize polygons and the respective diagonals; how they see an algebraic model in order to generalize the total number of diagonals of n sided polygon; and how they handle the generalized formula to find the number of diagonals of any polygon.

For data analysis it was clustered the student's written responses according to their academic and professional background. The background of the seven selected students is presented in Table 3.

Table 3: Academic and professional background of seven cases studies

Number of students/Category	*Academic background*	*Professional background*
2 / (I)	Grade 12	None
1 / (II)	Grade 12	Grade 7+ 2 years lower primary school teacher training course
1 / (III)	Grade 12	Grade 10+2 years upper primary school teacher training course
2 / (V)	Grade 10	Grade 10+2 years upper primary school teacher training
1 / (VI)	Grade 10	Grade 10+3 years technical college teacher training

There, the results showed that even though the students had been attending Analytic Geometry, they did not access that knowledge to solve these tasks. Task 1 could be solved using knowledge from either Euclidean or Analytic Geometry. Indeed, the results also showed that three of the students (one student each from category I, II, and VI) accessed algebraic and geometric knowledge but they used it inappropriately for Task 1. Here is an example of the Student I´s (student of category I) solution of Task 1 (Fig 1). He correctly sketched the geometric figure and correctly conjectured about the segments DE and BC. He wrote the segments DE and BC are parallel to each other. However, he used this conjecture as a given instead of a conclusion of the theorem. During the process of the demonstration of this theorem he correctly wrote the (algebraic) relations $AB = 2AD$ and $AC = 2AE$, nevertheless he did not use these relations for the solution of the task. He used the congruency of angles to show the conjecture, however, the justification of the congruency was incorrect. For example he wrote $< D \cong < B$ because they were opposite angles.

Besides, two of them (one student each from category II and VI) after identification of the given (premise) and the conclusion, i.e. the relation $p \Rightarrow q$, they used a property of the conclusion to prove the conclusion. Here we refer to the example of Student II´s solution as shown in Fig 2. Student II correctly conjectured $DE // BC$. He also correctly wrote the (algebraic) relations $DC = \frac{1}{2} AC$

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and $AE = \frac{1}{2} AB$. However, he did not use them for proving. Besides, he used the property of parallelism to show the congruency of certain angles that led him nowhere.

Figure 1: Student of Category I- Task 1

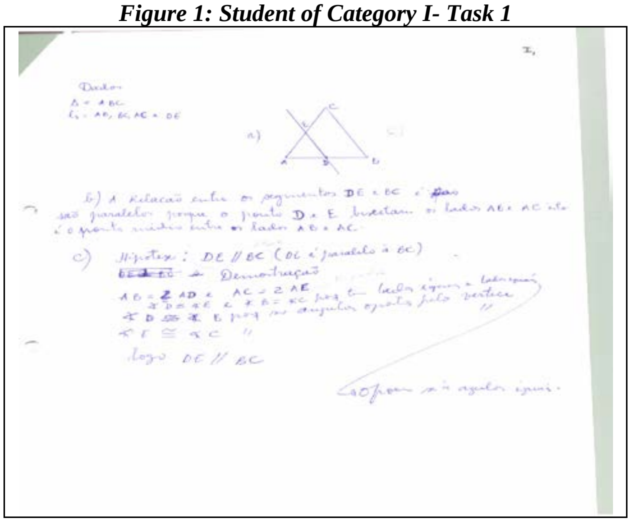
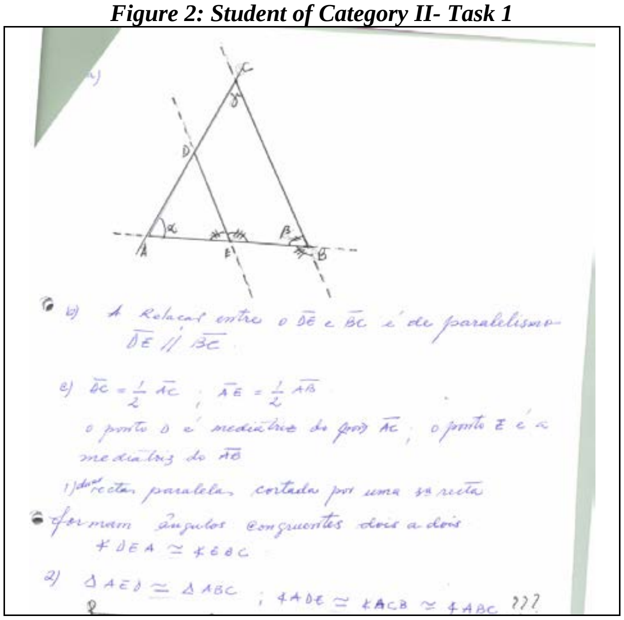


Figure 2: Student of Category II- Task 1



The other Student I and one of the students of category V accessed some algebraic and geometric concepts, but they could not use them to solve the tasks. Student V´s solution to Task 1 serves as an example to show this.

Although Student V correctly sketched the figure and identified the conjecture, he could not carry on the demonstration of the theorem. Student III and one of the students of category V accessed all key algebraic and geometric concepts and they used them successfully for Task 1.

Student III conjectured $BC = 2DE$. He used an auxiliary theorem that if a line bisects two sides of a triangle, then it is parallel to the third side to show that the triangles ADE and ABC are similar to each other. Then, using the criterion of proportion for similar triangles he came up with the conclusion. An example of the solution of Student III is given in Fig 4.

Figure 3: Student of Category V- Task 1

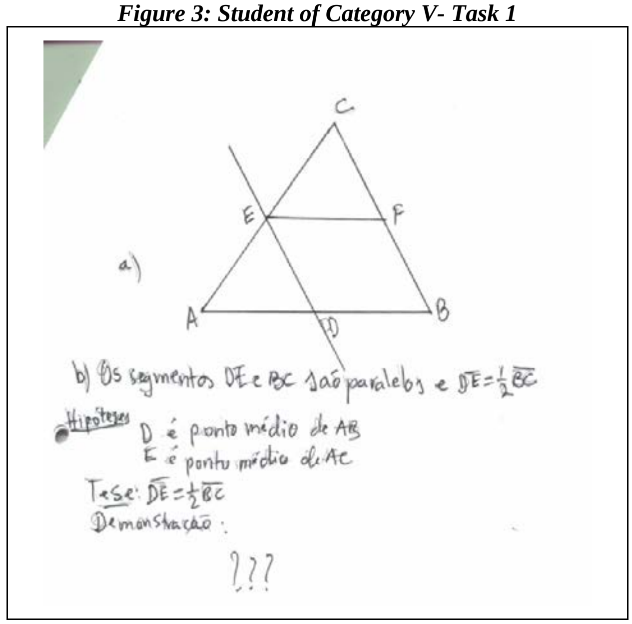
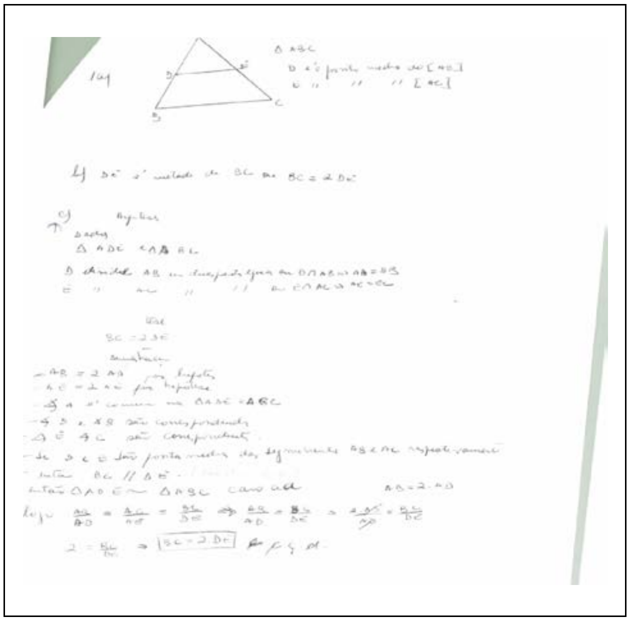


Figure 4: Student of Category III- Task 1



Although all students were aware that they had to find a pattern and then generalize with a formula, they faced difficulties in identifying patterns, making generalizations, and reasoning processes in Task 2.

A solution of one of the students is presented to illustrate this point as presented in Fig 5. This student (see result in Fig 5) faced difficulties in the process of deducing a formula.

It seems that he started off with a triangle, some squares, some pentagons, some hexagons to see whether he could get a pattern and finally a formula. It seems, however, that he was unsuccessful, and he tried to draw a 70- sided polygon to get the number of its diagonals from the pictorial representation. Meanwhile, it seems that he abandoned this approach as we can see that he tried to erase the solution.

Further on (Fig 6) we can see how he continued his solution using a similar approach but with some systematization. However, he also was unsuccessful. It seems that one of the difficulties the student faced was in the stating the correct definitions of a diagonal and of a polygon. He wrote that a 2-sided "polygon" had 2 diagonals; a triangle possessed 3 diagonals and so forth.

Figure 5: Solution 1 of Task 2

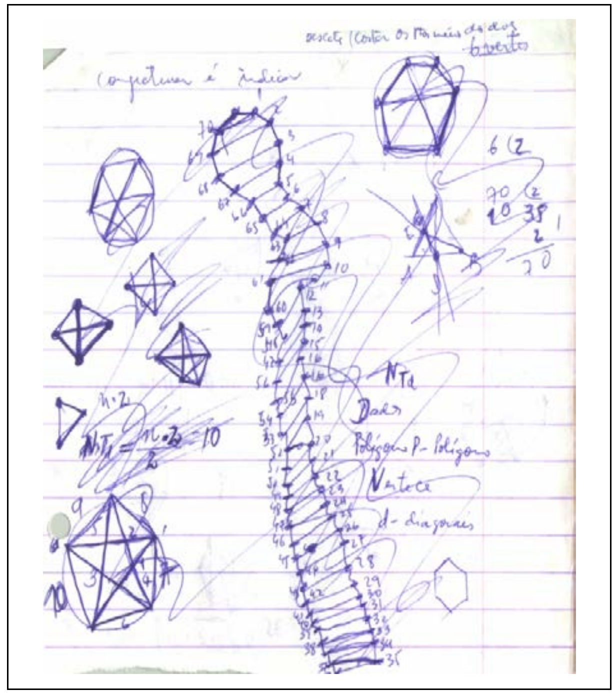
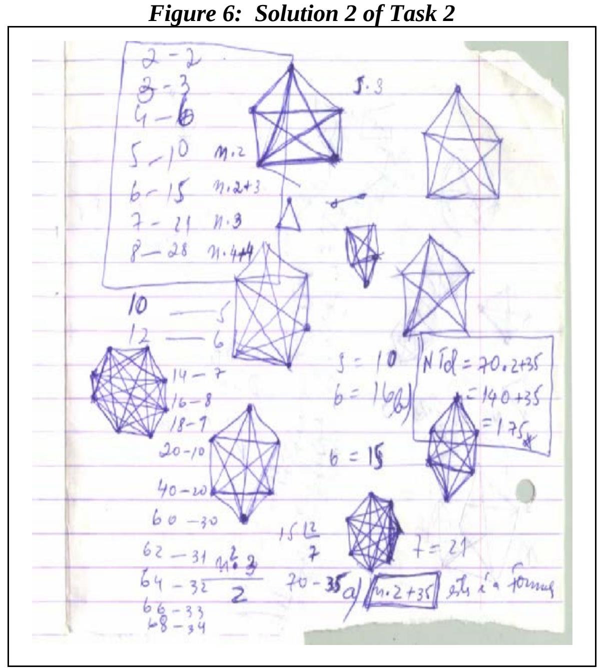


Figure 6: Solution 2 of Task 2



All results presented above are summarized in the following Table 4.

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Table 4: Results of the pilot test in accessing and using knowledge of a sample

Result / Group	*Accessed all key concepts and used them appropriately for solving Task 1*	*Accessed most key concepts and did not use them appropriately for solving Task 1*	*Accessed some concepts and did not use them for solving Tasks 1 and 2*	*Accessed some concepts and did not use them for solving Task 2*
Group I	None	1	1	2
Group II	None	1	None	1
Group III	1	None	None	1
Group V	1	None	1	2
Group VI	None	1	None	1
Total	**2**	**3**	**2**	**7**

4. Discussion and Conclusions

The students came from different academic and professional backgrounds. This fact partially seems to account for different categories of accessing and use of algebraic and geometric knowledge in the tasks amongst the students. Although these students passed Euclidean Geometry and were attending Analytic Geometry, they showed difficulties in accessing and using knowledge in both domains when solving the tasks of the pilot test. These difficulties seem to be caused by lack of development of some indicators of algebraic and geometric thinking, namely, making relations, identifying patterns, making generalizations and reason. Therefore, tasks were designed in collaboration with the lecturers of these disciplines mostly assessing these indicators of algebraic and geometric thinking for the main study.

In the following, are described some issues emerging from the analysis of the students´ written responses on the written test:

Firstly, students entering at UP enrolled in the course of Mathematics Teacher Training Program came from different academic and professional backgrounds. Accordingly, they experienced different geometry teaching and learning contexts.

> *Insight 1: The differing contexts of teaching and learning develop students academically differently.*

Secondly, although UP may create a homogeneous milieu for teaching and learning of geometry, it seems that students still face difficulties in accessing and using algebraic and geometric knowledge at least in the first year.

> *Insight 2: The University mathematics education setting, however suitably packaged, may come too late in the development of students' capabilities to use algebraic thinking in geometry.*

Thirdly, it seems that students needed to develop algebraic and geometric thinking during the course through different pedagogical activities (e.g. construct appropriate tasks where the indicators of algebraic and geometric thinking are assessed).

> *Insight 3: In order to develop algebraic and geometric thinking during a course, a range (rather than one form) of powerful activities are needed.*

Thus, as a conclusion, we can say that this study enabled researcher to rearrange some aspects concerned with appropriateness of the tasks for exploring the interplay between algebraic and geometric thinking for the next stages (the Main Study – Euclidean Geometry Course and the Main Study – Analytic Geometry Course) as well as to find out the first-year students' academic and professional background at the beginning of their degree at Pedagogic University (UP). From the students' results I concluded that the differing contexts of teaching and learning developed students academically differently before entering UP. Although UP has been making efforts to create a homogeneous *milieu* for teaching and learning of geometry, it seemed that students still faced difficulties in accessing and using algebraic and geometric knowledge at least in the first year. These results allowed me to be open to other factors, besides the conceptual, such as contextual and structural issues which may influence the students' understanding and their use of algebraic knowledge and thinking for solving geometric tasks.

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L. W. San: - PhD (2010) and MSc (1997) in Mathematics Education at the University of the Witwatersrand, South Africa. Lecturer at the University Eduardo Mondlane, since 1985; 2 articles published in International Journals, 7 scientific papers presented in National/International Mathematics Conferences and Congresses.