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First-Year University Students´ Algebraic Thinking and its Relationship to their Geometric Conceptual Understanding

Luis Weng San, Bhangy Cassy and Ribas Guambe

Dept. of Mathematics and Informatics, University Eduardo Mondlane
Maputo, Mozambique

Abstract

This study proposes to investigate how first-year university students bring their knowledge and thinking of Algebra in understanding and working with Geometry. The study explores how students connect and use algebraic and geometric concepts and also investigates whether this connection promotes their conceptual understanding and their problem solving performance in geometry. Data were collected, over one academic year, using written tests administrated to students as pre- and post-tests and achievement tests. Interviews based on students written responses to the tests were performed involving the eight selected students to get deeper on their algebraic and geometric knowledge base, connectedness and strategies when involved in mathematics problem solving, in light with a conceptual model that can relates the algebraic thinking with a geometrical understanding, under the framework on learning and transfer and, whether they accessed and utilized their intellectual resources in problems situations where those resources might be relevant.

Keywords: *Algebra, geometry, algebraic thinking, geometrical understanding, problem solving.*

1. Introduction

Worldwide, it is recognized that teaching geometry, particularly producing geometry proofs, is more complex and often less successful than teaching numerical operations or elementary algebra, and according to authors as [3] and [2], this difficulty, possibly, arises due to the underlying cognitive complexity of geometrical activity. They also found in their studies that the high school students involved in their studies faced a lot of difficulties in solving geometric problems concerning with the interplay between geometric and analytic approaches, possibly due to a weak grasp of geometric concepts from previous grades, and they also concluded that several factors have contributed to the inadequate understanding and performance of students in geometry. Several factors have been pointed out as affecting the ways in which students understand and perform in mathematics tasks, and among these factors may be of contextual and conceptual in nature. These may be related to the nature of the discipline of geometry itself, and the school mathematics contents and the nature of geometry teaching in Mozambique. However, what is less clear, particularly in the context of mathematics education in Mozambique, is how students work with geometry problems, what algebraic knowledge, if any, they bring to the solving of these problems, and how knowledge of algebra and algebraic ways of thinking promote or hinder success in geometry problem-solving processes.

On one hand, [2] also stated that little or no research have been carried out on the influence of algebra in the geometry curriculum, in particular, algebraic thinking in geometry has not been explored enough, in this study we attempt to make a contribution to enhance the learning of mathematics and in particular the linkage between the knowledge of algebra to understand geometry. On the other hand, [6] supports this conjecture when he asserts that "Descartes was disturbed by the fact that every proof in Euclidean geometry called for some new, often ingenious, approach [6:308]". Also [13] describes geometry as a dual subject in its essence: "First there is the visual, self-contained, synthetic side, which seems intuitively natural; then the algebraic, analytic side, which takes over when intuition fails and integrates geometry into the larger world of mathematics" [13:105]. The dual nature of geometry makes the subject extremely rich. It allows the interplay between the mathematical language (e.g. algebra) and the language of pictures, between the synthetic approach (where at each step what you say has a meaning in terms of figure) and the analytic approach (using coordinates to transfer the questions to a numeric or algebraic framework).

Therefore, it seems that a way to improve students` understanding of geometric concepts and problem solving performance in geometry might be the use of algebra. Algebra may promote flexible and abstract image schemata in students` mind. This can facilitate students to move from particular concrete images (Euclidean Geometry) to abstract logical structures (e.g. analytic geometry). On the other hand, [12] contends that the bridge between Euclidean geometry and abstract algebra is the analytic geometry invented by Fermat and Descartes around 1630 which is a method devoted to a thorough going application of algebra to geometry and geometry to algebra. And, according to [8], in order to use the power of algebra as an aid into geometric understanding, one needs to look into what constitutes algebraic and geometric understanding and how algebraic understanding enhances geometric understanding. Thus, a way of looking into algebraic and geometric understanding, it is certainly, important to look at mathematical understanding from a general perspective as cumulative structuring; as social process and as forms of knowing. So, for an individual to understand algebra or geometry should be aware of the way these branches of mathematics are built; and thus, should be involved in a social processes through discursive practice in particular settings (e.g. school) and then should develop 'fluency' of the language of algebra or geometry.

Although, there have been very few empirical studies that have investigated the link between algebraic thinking and geometric understanding. For instance, the study by [11] tried to look what an individual need to understand, in order to develop competency in a particular mathematical domain, for example in geometry and this study is quite important for our research as it pointed out the need to think about how cognitive processes underlying geometric conceptual understanding should be acquired and nurtured as interconnected processes in practice (e. g. in the teaching-learning process, and elaboration of suitable geometric tasks). It raised an important question about what it means to `understand` particular mathematical ideas (e.g. critical algebraic and geometric concepts and their relationships) and what it takes to be able to access and use algebraic thinking and knowledge in order to understand and perform well in geometry.

However, [7] investigating the relationship between success in problem solving and the quality of the knowledge connections developed by students in geometry, found that the organizational quality of students' geometric knowledge is associated with better problem-solving performance, and that, an individual with a high level of organized knowledge could retrieve more knowledge spontaneously and activate or establish more links among given knowledge schemas and related information than would not be possible otherwise. On the other hand, [2], in a study involving secondary students' use of algebraic thinking stated that the use of symbols and algebraic relations as well as the use of different forms of representations, and the use of patterns and generalizations in geometry and their related conceptual difficulties, attempted to find out the nature of the algebraic thinking these students used in their geometry course and whether teachers encouraged the use of this type of thinking in the class, and thus, arriving to the conclusion that students used algebraic thinking in solving problems in geometry, and that algebraic thinking is strongly connected to the thinking in geometry.

Other researchers, for example, [5:40] referring to results of a study carried out in the contest of Mozambique, stated that poor performance by Grade 5 pupils in the national mathematics test specifically in geometry is primarily due to "the philosophy which underlies the mathematics syllabi in Mozambique". Thus, contends that the school mathematics syllabi in Mozambique (which is an African country) are a copy of those from some European countries, which are deprived of a concrete African country context. Some syllabi contents are taught before pupils have reached a sufficient cognitive development level or before they have consolidated previous concepts needed to understand new concepts and this author also acknowledged that the existence of a great deal of untrained teachers can also influence the poor performance of pupils in mathematics.

The above discussion points to a critical relationship between algebraic thinking and geometric understanding, which are essential elements of mathematical understanding. Accordingly, the schema below (Figure 1) that represents a *Conceptual Model for Algebraic Thinking in Geometrical Understanding* is an attempt to highlight the connections between algebraic thinking and geometrical understanding [1], [13].

As can be seen, the discussion above highlighted the importance of the understanding of algebra to work in geometry and allow us conclude also that students might have better preparation in algebra prior to joining the geometry class. Thus this study proposes to focus on the former factor, how students bring algebraic knowledge and thinking as they work with geometry at university level.

So, the study aims to explore what kinds of meanings the Mozambican first-year university students show for different critical algebraic and geometric concepts that they are working with. With this regard, the key issue of the study proposes to investigate how first-year university students at Pedagogic University (UP) in Maputo-Mozambique bring their knowledge and thinking of algebra in understanding and working with geometry. The study also explores how these students connect and use algebraic and geometric concepts and investigates how and to what extent those students make use of algebraic knowledge and thinking as they work on geometry problems at university level and whether this connection

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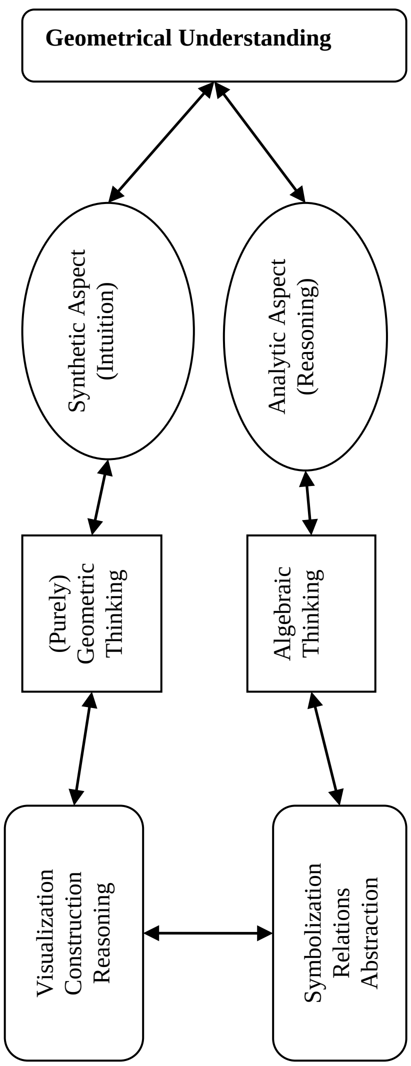
promotes students' conceptual understanding and problem solving performance in geometry.

Figure1: Conceptual model for Algebraic Thinking in Geometrical Understanding



2. Material and Methods

Hence, the aim of this study could best be approached through a qualitative research design that can account for multiple perspectives and provides space for explorations, with the idea to collect extensive data with an open mind, the methods used are: Nonparticipant observation, meetings and interviews, and document and artifact analysis (Euclidean and Analytic Geometry syllabuses, students´ written responses to tests, concept maps, and elaboration tasks). As described below, these data collection methods were complementary since they were intended to produce data that would achieve a holistic understanding of the questions under investigation. And thus, as the study progressed, data were continually examined for emerging patterns and insights. This is a type of qualitative methodology called 'Grounded Theory', [10] through case studies [15].

The **nonparticipant observation** is one of several methods for collecting data considered to be relatively unobtrusive [10]. The study of [10] describe two types of nonparticipant observation: disguised field observation and naturalistic field experience. For this study I used the latter type as I intended to observe the classroom context as it unfolds naturally taking care of the subjects' circumstances. So, during the observation the role of the researcher was attempting to video record the most important instances of the lectures, such as lecturer´s and students´ actions and lecturer– students' interactions).

The **semi-structured interviews** were used involving the target students and the lecturer teaching Euclidean Geometry, with the intention to supplement the data obtained from various other sources. This technique corroborate with [10] arguing that a researcher, particularly one who will be in the setting for a considerable period of time… may choose to conduct a series of relatively unstructured interviews that seem more like conversations with the respondents and that topics would be discussed and explored in a somewhat loose but probing manner. So, for this study we felt the need to return periodically to the respondents to find out more in depth what they did in their written responses at different stages of the courses to focus "on questions further or to triangulate with other data". Those students were individually interviewed using Interview Tasks 1 (Free Recall and Hinting Tasks on their pre-test responses), and Interviews Task 2 (Elaboration and Concept Mapping Tasks). Thus, the interviews were audio recorded and later transcribed. The students should respond to the questions regarding their solutions in several written tests administered to them, and several times they produced additional ideas to the solutions. Each interview lasted about one hour to one and half hour depending on the interviewees' responses.

The **Document and Artifact Analysis:** As defined in the study by [10:1058], **artifacts** of interest to researchers "are things that people make and do". In this study, there were collected course syllabuses, achievement tests, instructional plans (exercise sheets), and students´ written responses to achievement tests and interviews answers. So, according to [10] [**documents and**] **artifacts** may help illuminate research questions when subjected to microanalysis. Thus, for this research, the purpose to collect such materials was to enrich the discussion of the research findings.

3. Results and Discussion

3.1 General Findings

In order to develop task-based instruments to be used in the study, a 55-minute pilot test was administered to 28 first-year students at Pedagogic University (UP). Those students had completed the Euclidean geometry at the university and were attending the analytic geometry course.

So, it was chosen a sample of test responses for analysis and the responses were clustered according to students' academic and professional background: *(i) two students have completed grade 12; (ii) two have completed grade 12 and a primary school teacher training course and a secondary school teacher training course respectively; iii) two have completed grade 10 and a secondary teacher training course; and (iv) one has completed grade 10 and a teacher training course for technical college*. This test aimed to explore students´ algebraic and geometric knowledge and thinking and their ability to access and use algebraic knowledge in geometry problem solving. Preliminary results show that even though the students had been attending analytic geometry, they did not access that knowledge to solve the Euclidean geometry tasks. These tasks could be solved using knowledge from either Euclidean or analytic geometry. Besides, four of the students were able to access algebraic and geometric knowledge but they used it inappropriately. One of them could not even access knowledge at all. Two could access algebraic and geometric knowledge and used it successfully in task 1. All students faced difficulties in identifying "patterns" and making "generalizations" and "reasoning" processes mainly in Task 2.

For the main study on Euclidean Geometry, a 60- minute pre-test was administered in the start of the Euclidean Geometry course to 32 first-year university students. These students constituted the only first-year university class pursuing secondary and high school pre-service mathematics teacher course. This test assessed the students' proficiency in school geometry and algebra when they enter the university. The pre-test was jointly constructed with the lecturer of Euclidean Geometry at the Pedagogic University (UP). Students' responses to this pre-test were examined and analyzed and 14 target students (8 low achievers and 6 high achievers) to follow-up during the study were identified. These students were individually interviewed using Interview Tasks 1 (Free Recall and Hinting Tasks on their pre- test responses).

After analyzing the target students' responses in the written tests, eight students, enrolling the Euclidean Geometry subjects at the university, were selected for the follow up in the study, and as complementary method to the data analysis, classroom observation was done. A Concept Mapping Task was used in the form of constructing true propositions using the concepts presented in the test's tasks. So, the data that was collected over one academic year considered the students' written responses to pre- and post-tests, achievement tests, interviews based on their responses the written tests, concept maps, classroom observations on selected geometry topics, and interviews with the course lecturers.

Therefore, the results on the theoretical underpinning and data analysis on the pre-test responses (algebraic and geometric knowledge base, connectedness and strategies) of the eight target students in light of the conceptual model suggested (algebraic thinking in geometrical understanding) was performed, under the framework on learning and transfer drawn up from [9], to see whether they accessed and utilized their intellectual resources in problems situations where those resources might be relevant.

For instance, for analysis it was taken two questions of a test with 4 Euclidean Geometry tasks that was set up for the purpose of this study and validated by sharing with the lecturer who was teaching Euclidean Geometry to those students at Pedagogic University in Maputo (UP). The chosen tasks were task 3 and task 4. The whole test comprised four Euclidean Geometry problems and was written in Portuguese language as it is the medium of instruction and the official language in Mozambique).

3.2 Task 3 Analysis

Thus, taking on analysis of Task 3 (bellow), from a test consisted of four tasks (see Appendix), the results from the students´ written responses showed as described below:

Task 3*: - Observe the cube [ABCDEFGH]: - Can you fit any additional pyramid(s) congruent to [ABCDH] into the cube? If 'yes', draw it!*

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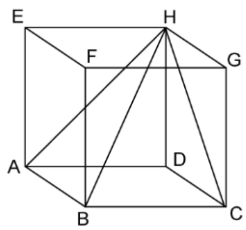

When the analysis was performed on **Task 3**, it was observed that only one of the eight selected students (Student 1) used key algebraic concepts to solve it. This student compared the volume formulae of a cube, ($V = A_b.h$, where, A_b = Area of the base, and h = altitude) and, the volume formulae of a pyramid ($V = \frac{A_b.h}{3}$), to show that at most three congruent pyramids fitted into the cube. The student showed it with a concrete cube of side 3 cm. He numerically concluded that the volume of a cube was three times as big as the volume of a pyramid with the same base area and altitude. He wrote down pyramids [EFGHB] and [BCGFE]. He only constructed pyramid [EFGHB] that leans on the original pyramid [ABCDH] by the segment line BH. However, pyramids [EFGHB] and [BCGFE] intersect each other, though they do not intersect the original pyramid. The three pyramids do not intersect each other and fit into the cube are [ABCDH] (the original), [ABFEH] and [BCGFH]. [In his strategy the student wrote: At most three congruent pyramids fit in the cube with 27 cm^3 as the volume of a pyramid is 9 cm^3 and it has the same altitude as the altitude of the cube].

On the other hand, it was found another student (Student 4) who only wrote down that more than three congruent pyramids fitted into the cube beside the original that is a total of four pyramids. He only wrote down pyramids [ABEFC], [ABCDE] and [EFGHB] and he did not construct any pyramid. However, it seems that he possessed a mental picture of a pyramid because he gave an algebraic representation of a pyramid. These pyramids intersect each other. His solution was as follow: - "*The student wrote: Yes, beside the original pyramid we can get three more pyramids, namely [ABEFC], [ABCDE] and [EFGHB]*".

Following the analysis and interpretation of the students' responses it was also found another student (Student 7) whose response asserted that some pyramids congruent to the original could fit into the cube. To justify his response, he constructed three pyramids, and according to his solution it seems that he concluded that at least four pyramids might fit into the cube, as he indicated pyramids [ADHEC], [DCGHA] and [ABFEG]. At least pyramid [ADHEC] intersects pyramid [DCGHA] and with the original pyramid.

Therefore, Student 8 wrote down three congruent pyramids to the original fitted into the cube, namely [FGHEA], [EFBAG] and [CDHGB]. It seems that he assumed four pyramids fitted into the cube, namely the three pyramids he suggested and the original. At least Pyramid [FGHEA] intersects pyramid [EFBAG]. He did not construct any pyramid, writing his response only as, "*Yes, [FGHEA], [EFBAG] and [CDHGB]*". Similar reasoning was seen in Students 4 and 5 responses.

Attempting to discuss above results on Task 3, Table 1 was constructed to show the students´ written responses categorized. The codes were produced according to the aim of the study, namely to see whether the students appropriately utilized algebraic thinking and knowledge in understanding and working with geometry.

Table 1: The analysis summary of the students` solution to Task 3

Student	***Algebraic Thinking***				***Geometric Thinking***			***Strategy***	
	S	**F**	**R**	**AR**	**V**	**C**	**R_s**	**D**	**E**
1	+	+	+	+	±	±	±	+	±
2	+	-	-	+	±	±	-	-	±
3	+	-	-	+	±	±	-	-	±
4	+	-	-	+	±	-	-	-	±
5	+	-	-	+	±	-	-	-	±
6	+	-	-	+	±	±	-	-	±
7	+	-	-	+	±	±	-	-	±
8	+	-	-	+	±	-	-	-	±

The explanation of the codes is as follows:

"S" = Symbolization (e.g. parameters and acronyms)

"F" = Formulae

"R" = Relations (between two formulae)

"AR" = Algebraic representation of the mathematical entity (e.g. the symbol used for naming a pyramid)

"V" = Visualization

"C" = Construction (Pictorial representation of the mathematical entity)

"R_s" = Reasoning

"D"	=	Deductive approach (mainly used in analytic strategies in geometry)
"E"	=	Empirical approach (mainly used in synthetic strategies in geometry)
"+"	=	Good use of the feature
"$\pm$"	=	Reasonable use of the feature
"-"	=	Failure or not use of the feature

There, we can see that in the analyzed task 3, all students accessed algebraic and geometric thinking. However, only one student within the group of eight selected students (the Student 1) accessed key algebraic concepts to partially solve this task. This student used the volume formula of a cube and of a pyramid, and the relationship between these two volume formulae. Here, he realized that at most three congruent pyramids including the original fitted into the cube, as the volume of the given pyramid is one third of the volume of the cube with the same altitude and base area, though it did not help the student to visualize and construct the three congruent pyramids in the cube. The other students only used some algebraic symbols to represent pyramids, congruency and so on. This algebraic thinking seemed not sufficient to prompt connections with geometric thinking and solve the problem completely. However, it seemed that algebraic thinking partially aided geometric thinking to get more insight into the task. All students were aware of the key geometric concept "pyramid". We can infer it through the algebraic and pictorial representation of a pyramid they provided.

From Table 1, above, we can conclude that all students but one used an empirical approach (as a means of development of intuition) that is according to their responses it seemed that they were guessing how many pyramids could fit into the cube through trial and errors using visualization and construction processes- the inference of a general law from particular instances [14]. In turn, Student 1 used a deductive approach (a means of discovery) - the inferring of particular instances from a general law [14]. He used the volume formulae and relations to conclude that at most three congruent pyramids fitted into the cube. Both approaches used separately it seems not to help students to succeed in geometric problem solving environment. [11] Observed that unless students learn to take advantage of both approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge. In turn, [4] corroborates this view explaining that reasoning takes place when by experimentation [e.g. construction by ruler and compass or geometrical software] and inductive generalization [by visualization processes], one extends her geometrical knowledge about shapes and relations and extends her "vocabulary" of legitimate ways of reasoning. Deductive reasoning [dependent exclusively on the corpus of propositions- definitions, axioms, and theorems] then, becomes a vehicle for understanding and explaining why and inductively discovered conjecture might hold.

3.2 Task 4 Analysis

Another corroborating example of a similar result is Task 4 as follows from the test:

Task 4: - *A quadrilateral possesses two diagonals, a pentagon five diagonals, and a hexagon possesses... diagonals. Determine an algebraic expression for the total number of diagonals of a polygon with n sides.*

In this task 4, students were required to provide an algebraic generalization. None of the eight students provided an algebraic expression to determine the total number of diagonals of a polygon. Although, only Student 2 presented some pattern in the sequence, to determine the total number of diagonals of a polygon, using the following strategy:

Number of sides of the polygon	*Total number of diagonals*
4	2
5	2+3=5
6	5+4=9
7	9+5=14

Of course, this strategy can be efficient only if it's known the total number of diagonals of the previous polygon. However, it does not allow producing a general formula for this sequence of numbers which relates the number of the polygon sides and the number of its diagonals. The student did not present a pictorial representation in his response.

On the other hand, Student 7 tried to relate the number of sides of a polygon and the number of its diagonals and labeled as n and a_n respectively. Though, because of this result, Student 7 concluded that, it was not possible to find a general algebraic expression to determine the total number of diagonals of any polygon as each polygon possessed properties different from the other regarding the number of its diagonals. There, the student wrote: "*It is impossible to determine an exact formula for the determination of the number of diagonals of an n- sided polygon… Each polygon possesses its own properties regarding the number of its diagonals*".

The strategy used by Student 7, on task 4, is as shown below:

Number of sides of the polygon – "n"	***Total number of diagonals – "a_n"***
4	$2n-6$
5	$2n-5$
6	$2n-3$
7	$2n-2$

On the other hand, Student 5 inductively tried to find the total number of diagonals of some polygons starting from a quadrilateral up to an octagon. He just presented a picture of a heptagon. He correctly wrote down the total number of diagonals from a quadrilateral to an octagon. He tried to get a generalizing formula of the total number of diagonals of any polygon, but he could not carry on further his solution. Another student, Student 1 after determining the number of diagonals of a hexagon (9 diagonals) he carried on in determining the number of diagonals of a heptagon (10 diagonals). He missed to count the remaining 4 diagonals, as the total diagonal number of a heptagon is 14. He drew an octagon but he did not count its diagonals. Besides, he missed to draw 3 of its diagonals. Other students, as Students 3 and Student 4, concluded that it was not possible to find a formula that can be used to find the total number of diagonals of any polygon, after drawing up 3 diagonal in a hexagon. On the other side, Student 6 only constructed the hexagon and its diagonals and he did not conclude anything about the total number of its' diagonals. At the end, one student (Student 8) left the Task without response. So, looking to the students' responses to Task 4, a summary of the analysis on the students' written responses is given in the following Table 2.

Table 2: The analysis summary of the students´ solution to Task 4

Student - Approach	***Synthetic(Intuition)***		***Analytic(Reasoning)***		
	Visuali-zation	***Construc-tion***	***Relation***	***Patter-ning***	***Generali-zation***
1	±	±	-	-	-
2	-	-	+	+	-
3	-	-	-	-	-
4	-	-	-	-	-
5	+	+	-	-	-
6	-	+	-	-	-
7	+	+	±	-	-
8	-	-	-	-	-

The codes "+", "±", and "-" used in Table 2 have the same meaning as the codes used in Table 1.

This Task 4 allows the *interplay* between synthetic and analytic approaches, and accordingly between algebraic and (purely) geometric thinking. As can be seen in Table 2, in general, the students faced difficulties in accessing key algebraic and geometric concepts to solve the Task 4. There can also be seen that Student 2 made a relation between the numbers of diagonals of two neighboring polygons and also discovered a pattern. Nevertheless this pattern did not lead to a general formula for the number of diagonals of an *n-sided polygon*. In turn, Student 7 correctly constructed and visualized the hexagon and its diagonals, but it seems that he used a pattern to calculate the number of diagonals of a heptagon which did not work. He related the number of sides and diagonals of a polygon and constructed some formulae which did not lead to a general formula.

4. Conclusions

From the analysis above we can see that algebraic thinking aided geometric thinking towards a partial solution of Task 3 (see Student 1 solution). For a total solution of this task it was additionally needed visualization and construction processes to present the three congruent pyramids in the cube. These results confirmed what [9] asserted that in order one to develop connectedness it requires one to possess key concepts and procedures (from different domains) which provide the glue that holds cognitive structures together so as to make representational links. Besides, these results corroborate a [11] finding that unless students learn to take advantage of both approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge. The use of the different properties of algebraic thinking (within analytic approach) in connection seems to aid students to partially understand some ideas or concepts in geometry (as referred to Student 1 strategy). On the contrary, using them separately seems to hinder understanding in geometry. Using the properties of geometric thinking (within synthetic approach) separately leads to weak understanding of geometric ideas or concepts. It seems that the interaction between deductive and empirical approaches (or between analytic and synthetic approaches) to geometry (unfortunately not to be noticed in these subjects) helps students to succeed in geometric problem solving environment [11]. On the other hand, looking to the nature of Task 4 that required the students to apply synthetic and analytical approaches (or empirical and deductive approaches), according to the results on Table 2, it seems that only a synthetic approach was not enough for solving this task. It also required an analytical approach. A student used some properties of analytical approach (relation and patterning), even though he was not able to produce a formula as a generalization of the total number of diagonals of an n- sided polygon.

Concluding we could notice that "profiting from the interaction between deductive and empirical approaches to geometry students may reap the benefits of their knowledge" [11].

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L. W. San: - PhD (2010) and MSc (1997) in Mathematics Education at the University of the Witwatersrand, South Africa. Lecturer at University Eduardo Mondlane,since 1985. 7 Scietific papers presented in National/International Mathematics Conferences and Congresses.

B. Cassy: - PhD (2002 & MSc (1998) at Wits University in South Africa. Professor of Mathematics, at University Eduardo Mondlane in Maputo, Mozambique. - 38 years teaching experience. Author and co-author of several papers in national and international journals as well as in national and international Conferences and Congresses. Several books and textbooks published for university and school level.

R. Guambe - PhD (2012, Lulea University of Technology, Sweden); - MSc (2004, Wits University in South Africa); - Mathematics Lecturer at the University Eduardo Mondlane in Maputo Mozambique. More than 35 years of teaching experience. Two papers in international journals and some other presented in National/International Mathematics Conferences and Congresses.