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Periodic Solution for nonlinear system of differential equations depending on the probability density function of gamma distribution

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Abstract . The numerical-analytic method were introduced by Samoilenko has been used to study the Periodic solution of a new nonlinear system of differential equations depending on the probability density function of gamma distribution. Also these investigations lead us to improving and extending the results of Samoilenko and extended his method .

Keywords . Numerical-analytic method existence, uniqueness, stability Periodic solution, probability density function of gamma dirstibution, nonlinear system of differential equations.

1. Introduction.

The theory of differential equations is one of the most fascinating and successful areas of mathematics. Its results help us to prove important theorems and provide the inspiration for many useful concepts in other areas of mathematics [1,6,7,10,12,13,14]. Many of the most powerful techniques used in the application of mathematics to other sciences and engineering are based on differential equations theory [4, 5, 6, 9, 1,17]. Because of its wide applicability, its blend of geometric and analytic concepts, and the simplicity of many of its results differential equations provides an excellent introduction to modern mathematics. Recent developments in differential equations theory [7, 8, 13, 15,18]. Many author create and develop numerical-analytic methods[2,3,4,8,9,10,11,12] and schemes to investigate periodic solution of integral equations describing many applications in mathematical and engineering field.

Butris [3] has been used the numerical-analytic method to study the periodic solution for nonlinear system of differential equation depending on of gamma distribution which has the form

$$\frac{dx}{dt} = (t,\gamma(t,\alpha),x) \qquad \cdots (1.1)$$

where $x \in D$, and D is the closure of bounded domain and connected in R^n.

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This study leads us to improving and extending Butris [3] results, to investigate the existence and approximation of Periodic solution for nonlinear system of differential equations depending on the probability density function of gamma distribution.

Consider the following system of differential equations depending on the probability density function of gamma distribution which has the form

$$\frac{dx}{dt} = f(t, \gamma(t, \alpha, \beta, \mu), x) \qquad \dots \quad (1.2)$$

where $x \in D$, all real t and D is the closure of bounded domain and connected in R^n. The vector function $f(t, \gamma(t, \alpha, \beta, \mu), x)$ is defined on the domain:

$$(t, \gamma(t, \alpha, \beta, \mu), x) \in R^1 \times [0, T] \times D = (-\infty, \infty) \times [0, T] \times D, \qquad \dots \ (1.3)$$

is periodic in t of period T and continuous in the totality of variables and satisfies the inequalities:

$$|f(t, \gamma(t, \alpha, \beta, \mu), x)| \le M|\gamma(t, \alpha, \beta, \mu)|\ , \quad |\gamma(t, \alpha, \beta, \mu)| \le M_\gamma, \qquad \dots (1.4)$$

$$|f(t, \gamma(t, \alpha, \beta, \mu), x_1) - (t, \gamma(t, \alpha, \beta, \mu), x_2)| \le K|\gamma(t, \alpha, \beta, \mu)|\,|x_1 - x_2| \qquad \dots \ (1.5)$$

for all $t \in R^1$ and $x, x_1, x_2 \in D$, where $M = (M_1, M_2, \cdots, M_n)$is a positive constant vector and $M_\gamma$is a positive constant . The general formula for the probability density function of gamma distribution is:-

$$\gamma(t, \alpha, \beta, \mu) = \frac{\left(\frac{t-\alpha}{\beta}\right)^{\mu-1} \exp\left(-\frac{t-\alpha}{\beta}\right)}{\Gamma(\mu)} \quad , \qquad t \ge \alpha \,, \qquad \dots (1.6)$$

where $T \le \frac{\left(\frac{t-\alpha}{\beta}\right)^{\mu-1} \exp\left(-\frac{t-\alpha}{\beta}\right)}{\Gamma(\mu)}, \alpha\,, \ \beta, \mu$ are a positive constants.

We define the non-empty sets as follows:

$$D_{\gamma f} = D - M M_\gamma \frac{T}{2} \qquad \dots . (1.7).$$

Furthermore, we suppose that the greatest Eigen value λ_{max} of the matrix

$\Lambda = M_\gamma\, K\frac{T}{2}$ does not exceed unity,

i.e. $\lambda_{max}(\Lambda) < 1$. (1.8).

Lemma1.1. Let $f(t)$ be a continuous vector function in the interval $0 \le t \le T$, then

$$\left|\int_0^t (f(s) - \frac{1}{T}\int_0^T f(s)ds)ds\right| \le \alpha(t) \max_{t\in[0,T]} |f(t)|,$$

where $\alpha(t) = 2t(1 - \frac{t}{T})$. (For the proof see [16]) .

By using lemma 1.1, we can state and prove the following lemma.

Lemma 1.2. Suppose that the function of the probability density $\gamma(t,\alpha,\beta,\mu)$is continuous on the interval $[0,T]$, then

$$\left|\int_0^t \gamma(s,\alpha,\beta,\mu) - \frac{1}{T}\int_0^T \gamma(s,\alpha,\beta,\mu)ds)ds\right| \le M_\gamma \alpha(t)$$

is hold for all values of α, β, μ.

Proof. Taking

$$\left|\int_0^t \gamma(s,\alpha,\beta,\mu) - \frac{1}{T}\int_0^T \gamma(s,\alpha,\beta,\mu)ds)ds\right| \le (1 - \frac{t}{T})\int_t^T |\gamma(s,\alpha,\beta,\mu)|ds + \frac{t}{T}\int_t^T |\gamma(s,\alpha,\beta,\mu)|\, ds$$

$$= \left(1 - \frac{t}{T}\right)\int_0^t M_\gamma\, ds + \frac{t}{T}\int_t^T M_\gamma\, ds$$

$$\le\; M_\gamma\left[\left(1 - \frac{t}{T}\right)t + \frac{t}{T}(T - t)\right] \;= \alpha(t)M_\gamma.$$

So that

$$\left|\int_0^t \gamma(t,\alpha,\beta,\mu) - \frac{1}{T}\int_0^T \gamma(t,\alpha,\beta,\mu)ds)ds\right| \le \alpha(t)\, M_\gamma \;\cdots (1.10)$$

for all $t \in [0, T]$ and $\alpha(t) \leq \frac{T}{2}$.

2. Approximate of solution.

The investigation of approximate periodic solution of (1.2) is formulated by the following theorem:

Theorem 2.1. If the system (1.2) satisfy the inequalities (1.4),(1.5) and the conditions (1.6),(1.7) has a continuous periodic solution $x = x(t, \gamma(t, \alpha, \beta, \mu), x_0)$, then the sequence of functions

$$x_{m+1}(t, \gamma(t, \alpha), x_0) = x_0 + \int_0^t [f(s, \gamma(s, \alpha, \beta, \mu), x_m(s, \gamma(s, \alpha, \beta, \mu), x_0)$$

$$-\frac{1}{T}\int_0^T (f(s, \gamma(s, \alpha, \beta, \mu), x_m(s, \gamma(s, \alpha, \beta, \mu), x_0)ds]ds$$

$$\cdots (2.1)$$

with

$$x_0(t, \gamma(t, \alpha, \beta, \mu), x_0) = x_0 , \qquad m = 0,1,2, \cdots$$

is uniformly convergent as $m \to \infty$ in the domain

$$(t, \gamma(t, \alpha, \beta, \mu), x_0) \in R^1 \times [0, T] \times D_{\gamma f} \qquad (2.2)$$

to the limit function $x^0(t, \gamma(t, \alpha), x_0)$ which is defined on the domain (2.2), and satisfying the system of integral equations

$$x(t, \gamma(t, \alpha, \beta, \mu), x_0) = x_0 + \int_0^t [f(s, \gamma(s, \alpha, \beta, \mu), x(s, \gamma(s, \alpha, \beta, \mu), x_0)$$

$$-\frac{1}{T}\int_0^T (f(s, \gamma(s, \alpha, \beta, \mu), x(s, \gamma(s, \alpha, \beta, \mu), x_0)ds]ds$$

$$\ldots (2.3)$$

which is a continuous periodic solution of (1.2) provided that:

$$|x^0(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_0| \le M\, M_\gamma\ \alpha(t) \qquad \dots\ (2.4)$$

and

$$|x^0(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)| \le \Lambda^m(E-\Lambda)^{-1}M\, M_\gamma\, \alpha(t) \qquad \dots\ (2.5)$$

for all $m \ge 1$ and $t \in R^1$, where E is the identity matrix.

Proof. By lemma 1.2 and by mathematical induction, we find that:

$$|x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_0| \le M\, M_\gamma\, \alpha(t) \qquad \cdots (2.8)$$

for all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

i.e. $x_m(t,\gamma(t,\alpha,\beta,\mu),x_0) \in D$, for all $t \in R^1$ and $x_0 \in \mathrm{D}_{\gamma f}$.

We claim that the sequence of functions (2.1) is uniformly convergent on the domain (2.2).

Suppose that the following inequality is true:

$$|x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_{m-1}(t,\gamma(t,\alpha,\beta,\mu),x_0)| \le M\, M_\gamma{}^{m}[K\frac{T}{2}]^{m-1}\alpha(t) \qquad (2.10)$$

for all $m \ge 1$.

Now, we shall prove the following:

$$|x_{m+1}(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)|$$

$$\le K[\left(1-\frac{t}{T}\right)\int_0^t M_\gamma\ \ |x_m(s,\gamma(s,\alpha,\beta,\mu),x_0)-x_{m-1}(s,\gamma(s,\alpha,\beta,\mu),x_0)|ds$$

$$+\frac{t}{T}\int_t^T M_\gamma\ \ |x_m(s,\gamma(s,\alpha,\beta,\mu,x_0)-x_{m-1}(s,\gamma(s,\alpha,\beta,\mu,x_0)|ds]$$

$$\leq \left(1-\frac{t}{T}\right)\int_0^t M_\gamma \ M\, M_\gamma{}^{\,m}\left[K\frac{T}{2}\right]^{m-1} \alpha(s)ds$$

$$+\frac{t}{T}\int_t^T M_\gamma \ M \ \ M_\gamma{}^{\,m}\left[K\frac{T}{2}\right]^{m-1} \alpha(s)ds$$

$$= \ M\, M_\gamma \left[M_\gamma \ K\frac{T}{2}\right]^{m} \alpha(t)$$

and hence

$$|x_{m+1}(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)| \leq MM_\gamma \ \left[M_\gamma\, K\frac{T}{2}\right]^{m} \alpha(t) \ (2.11)$$

for all $m \geq 0$.

From (2.11) we conclude that for any$k \geq 1$, we have the inequality

$$|x_{m+k}(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)| \leq \sum_{i=0}^{k-1} \Lambda^{m+i}\, M\, M_\gamma \ \ \alpha(t)$$

such that

$$|x_{m+k}(t,\gamma(t,\alpha,\beta,\mu),x_0)-\mathrm{x}_m(t,\gamma(t,\alpha,\beta,\mu),x_0)|$$

$$\leq \sum_{i=0}^{\infty}|x_{m+1+i}(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_{m+i}(t,\gamma(t,\alpha,\beta,\mu),x_0)|$$

$$\leq \sum_{i=0}^{\infty} MM_\gamma\, \alpha(t)\, \Lambda^{m+1+i}$$

$$\leq MM_\gamma \ \ \alpha(t)\, \Lambda^m \sum_{i=0}^{\infty} \Lambda^{i+1}$$

$$\leq MM_\gamma \ \ \alpha(t)\, \Lambda^m \,(E-\Lambda)^{-1}$$

So that

$$|x_{m+k}(t,\gamma(t,\alpha,\beta,\mu),x_0)-x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)| \leq \ \Lambda^m \,(E-\Lambda)^{-1} MM_\gamma \ \ \alpha(t) \qquad (2.12)$$

for all $k \geq 1$.

From (2.12) and the condition (1.9), we find that:

$$\lim_{m\to\infty} \Lambda^m = 0. \qquad (2.13)$$

Relations (2.12) and (2.13) prove the uniform convergence of the sequence of functions (2.1) on the domain (2.2).

Let

$$\lim_{m\to\infty} x_m(t,\gamma(t,\alpha,\beta,\mu),x_0) = x^0(t,\gamma(t,\alpha,\beta,\mu),x_0) \qquad \cdots (2.14)$$

Since the sequence of functions (2.2) is a periodic and continuous in t,γ,x , then the limiting function $x^0(t,\gamma(t,\alpha,\beta,\mu),x_0)$ is also periodic and continuous in t,γ,x.

Moreover, by Lemma 1.2 and inequality (2.12) the inequalities (2.4) and (2.5) are holds.

3 **Uniqueness of periodic solution.**

We have to show that $x(t,\gamma(t,\alpha,\beta,\mu),x_0)$ is a unique periodic solution of the

system (1.1).Assume that $r(t,\gamma(t,\alpha,\beta,\mu),x_0)$ is another periodic solution of the system (1.1),

i.e.

$$r(t,\gamma(s,\alpha,\beta,\mu),x_0) = x_0 + \int_0^t [f(s,\gamma(s,\alpha,\beta,\mu),r(s,\gamma(s,\alpha,\beta,\mu),x_0)$$

$$-\frac{1}{T}\int_0^T (f(s,\gamma(s,\alpha,\beta,\mu),r(s,\gamma(s,\alpha,\beta,\mu),x_0)ds]ds,$$

$$(2.15)$$

Theorem2. $With$ the hypotheses and all conditions of the theorem1.1, the periodic solution of 1.1is a unique continuous on the domain (1.2). Now, we prove that $x(t,\gamma(t,\alpha,\beta,\mu),x_0) = r(t,\gamma(t,\alpha,\beta,\mu),x_0)$ for all $x_0 \in D_{\gamma f}$ and to do this, we need to drive the following inequality, we need to drive the following inequality:

$$|r(t,\gamma(t,\alpha,\beta,\mu),x_0) - x(t,\gamma(t,\alpha,\beta,\mu),x_0)| \leq \Lambda^m (E-\Lambda)^{-1} M^* M_\gamma \ \alpha(t) \qquad (2.16)$$

$where\ M^* = \max_{x_0\in D_{\gamma f}} \left|f\big(t, r(t,\gamma(t,\alpha,\beta,\mu),x_0), x(t,\gamma(t,\alpha,\beta,\mu),x_0)\big)\right|$

Suppose that (2.16) is true for $m = k$, i.e.

$$|r(t,\gamma(t,\alpha,\beta,\mu),x_0) - x(t,\gamma(t,\alpha,\beta,\mu),x_0)| \le \Lambda^k\ (E-\Lambda)^{-1}M^*M_\gamma\ \alpha(t)$$

For all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

Then

$$|r(t,\gamma(t,\alpha,\beta,\mu),x_0) - x(t,\gamma(t,\alpha,\beta,\mu),x_0)|$$

$$\le K[\left(1-\frac{t}{T}\right)\int_0^t M_\gamma\ |r(s,\gamma(s,\alpha,\beta,\mu),x_0) - x(s,\gamma(s,\alpha,\beta,\mu),x_0)|ds$$

$$+\frac{t}{T}\int_t^T M_\gamma\ |r(s,\gamma(s,\alpha,\beta,\mu),x_0) - x(s,\gamma(s,\alpha,\beta,\mu),x_0)|ds]$$

$$\le K[\left(1-\frac{t}{T}\right)\int_0^t M_\gamma\ \Lambda^k(E-\Lambda)^{-1}M^*M_\gamma\ \alpha(s)ds$$

$$+\frac{t}{T}\int_t^T M_\alpha\ \Lambda^k(E-\Lambda)^{-1}M^*M_\gamma\ \alpha(s)ds] = \Lambda^{k+1}(E-\Lambda)^{-1}M^*M_\gamma\ \alpha(t).$$

By induction, inequality (2.16) is true for $m = 0,1,2,\cdots$,

and thus from (2.14) and (2.16), we have:

$$\lim_{m\to\infty}|r(t,\gamma(t,\alpha,\beta,\mu),x_0) - x_m(t,\gamma(t,\alpha,\beta,\mu),x_0)| = 0$$

and hence

$$\lim_{m\to\infty} x_m(t,\gamma(t,\alpha,\beta,\mu),x_0) = r(t,\gamma(t,\alpha,\beta,\mu),x_0)$$

By the relation (2.14), we get:

$$x(t,\gamma(t,\alpha,\beta,\mu),x_0) = r(t,\gamma(\gamma(t,\alpha,\beta,\mu)t,\alpha),x_0)$$

i.e. $x(t,\gamma(t,\alpha,\beta,\mu),x_0)$ is a unique continuous solution of (1.1) on the domain (1.2). ■

3. Existence of solution.

The problem of existence of a periodic solution of the system (1.1)

is uniquely connected with the existence of zeros of the function $\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)$ which has the form:

$$\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0) = \frac{1}{T}\int_0^T f(s,\gamma(s,\alpha,\beta,\mu),x^0(s,\gamma(t,\alpha,\beta,\mu),x_0)ds \ldots\ldots\ldots\ldots \cdots (3.1)$$

where $x^0(t,\gamma(t,\alpha),x_0)$ is the limiting function of the sequence of functions (2.1).

The function (3.1) can find only approximately, say by computing the

following functions:

$$\Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_0) = \frac{1}{T}\int_0^T f(s,\gamma(s,\alpha,\beta,\mu),x_m(s,\gamma(s,\alpha,\beta,\mu),x_0)ds,$$

m= 0,1,2,… $\cdots$ (3.2)

Theorem3.1. Let all assumptions and conditions of theorem 2.1 were given. Then the inequality:

$$|\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)-\Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_0)| \leq \Lambda^{m+1}(E-\Lambda)^{-1} M\, M_\gamma \cdots (3.3)$$

will be satisfied for all $m \geq 0, x_0 \in D_{\gamma f}$.

Proof. By the relations (3.1) and (3.2), the estimate

$$|\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)-\Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_0)|$$

$$\leq \frac{K}{T}\int_0^T |\gamma(s,\alpha,\beta,\mu)|\,|x^0(s,\gamma(s,\alpha,\beta,\mu),x_0)-x_m(s,\gamma(s,\alpha,\beta,\mu),x_0)|ds$$

$$\leq \frac{K}{T}\int_0^T M_\alpha\, \Lambda^m(E-\Lambda)^{-1}M\, M_\gamma\ \ \alpha(s)ds$$

$$= \Lambda^{m+1}(E-\Lambda)^{-1}M\, M_\gamma$$

Thus the inequality (3.2) is hold for all $m \geq 0$.

Next, we prove the following theorem taking into account at the inequality

(3.3) will be satisfied for all $m \geq 0$.

Theorem 3.2. If the system (1.1) satisfies the following condition:

(i)The sequence of functions (3.2) has an isolated singular point $x_0 = x^0$,
$\Delta_m(t, \gamma(t, \alpha, \beta, \mu), x^0) \equiv 0$, for some $t \in R^1$.
(ii)The index of this point is nonzero;
(iii)There exists a closed convex domain D_γ^* belonging to domain $D_{\gamma f}$ and possessing a unique singular point x^0 such that on it is boundary $\Gamma_{D_\gamma^*}$ the following inequality is holds

$$inf_{x_0 \in \Gamma_{D_\gamma^*}} \|\Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_0)\| \geq \|\Lambda^m (E - \Lambda)^{-1} M\, M_\gamma\| \quad ..(3.4)$$

Where $x_0 \in \Gamma_{D_\gamma^*}$ for all $m \geq 0$. Then the system (1.1) has a periodic solution
$x = x(t, \gamma(t, \alpha, \beta, \mu), x_0)$ for which $x(t, \gamma(t, \alpha, \beta, \mu), x_0)$ belongs to the domain D_γ^* .

Proof. By using the inequality (3.1) we can prove the theorem 3.1by the same proof of a theorem7.1[13].

Remark 3.1[16]. When $R^n = R^1$, i.e. when $x_0$is a scalar, the existence of aperiodic solution can be strengthens by giving up the requirement that the singular point shout be isolated, thus we have

Theorem3.3. Let the system (1.1) is defined on the interval $[a, b]$. Suppose that for $m \geq 0$, the function $\Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_0)$ defined according to formula (3.2) satisfies the inequalities:

$$\left.\begin{aligned} \min_{a+h \leq x_0 \leq b-h} \|\Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_0)\| \leq -\sigma_m \quad ; \\ \max_{a+h \leq x_0 \leq b-h} \|\Delta_m(\gamma(t, \alpha, \beta, \mu), x_0)\| \geq \sigma_m \quad . \end{aligned}\right\} \cdots (3.5)$$

Then the system (1.1) has a periodic solution $x = x(t, \gamma(t, \alpha) x_0)$ for which

$x_0 \in [a + h, b - h]$, where $h = \|M\, M_\gamma\| \frac{T}{2}$ and $\sigma_m = \|\Lambda^{m+1}(E - \Lambda)^{-1} M M_\gamma\|$.

Proof. Let x_1 and x_2 be any two points on the interval $[a, b]$ such that:

$$\left.\begin{aligned} \Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_1) = \min_{a+h \leq x_0 \leq b-h} \Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_0) \; ; \\ \Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_2) = \max_{a+h \leq x_0 \leq b-h} \Delta_m(t, \gamma(t, \alpha, \beta, \mu), x_0) \; . \end{aligned}\right\} \cdots (3.6)$$

Taking into account inequalities (3.3) and (3.5), we have

$\Delta(t,\gamma(t,\alpha,\beta,\mu),x_1)$

$$= \Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_1) + [\Delta(t,\gamma(t,\alpha,\beta,\mu),x_1) - \Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_1)]$$

$\Delta(t,\gamma(t,\alpha,\beta,\mu),x_2)$

$$= \Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_2) + [\Delta(t,\gamma(t,\alpha,\beta,\mu),x_2) - \Delta_m(t,\gamma(t,\alpha,\beta,\mu),x_2)]$$

…(3.7)

It follows from the inequalities (3.7) and the continuity of the function

$\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)$, that there exist an isolated singular point $x^0, x^0 \in [x_1,x_2]$, such that $\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0) \equiv 0$, this means that the system (1.1) has a periodic

continuous solution $x = x(t,\gamma(t,\alpha,\beta,\mu),x_0)$ for which $x_0 \in [a+h, b-h]$. ■.

4.Stability of solution.

In this section, we study the stability periodic solution of the system (1.2) by the following theorem

Theorem4.1. If the function $\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)$ is defined by

$\Delta: D_{\gamma f} \to R^n$,

$\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0) = \frac{1}{T}\int_0^T f(s,\gamma(t,\alpha,\beta,\mu),x^0(t,\gamma(s,\alpha,\beta,\mu),x_0)ds\ldots\ldots \cdots$ (3.8)

where$x^0(t,\gamma(t,\alpha),x_0)$ is a limit of the sequence of functions (2.1). Then the following inequalities are holds

$|\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)| \le M\, M_\gamma \cdots$ (3.8)

and

$$|\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0{}^1) - \Delta(t,\gamma(t,\alpha,\beta,\mu),x_0{}^2)| \le \frac{2}{T}\Lambda(E-\Lambda)^{-1}M_\gamma \cdots (3.9)$$

for all $x_0, x_0{}^1, x_0{}^2 \in D_{\gamma f}$.

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Proof. From the properties function $x^0(t,\gamma(t,\alpha,\beta,\mu,x_0)$ established theorem 2.1; it follows that function $\Delta(t,\gamma(t,\alpha),x_0)$ is continuous and bounded by $M\ M_\alpha$.

By using (3.7), we get:

$$|\Delta(t,\gamma(t,\alpha,\beta,\mu),{x_0}^1)-\Delta(t,\gamma(t,\alpha,\beta,\mu),{x_0}^2)|$$

$$=\left|\frac{1}{T}\int_0^T f(s,\gamma(s,\alpha,\beta,\mu),x^0(s,\gamma(s,\alpha,\beta,\mu),{x_0}^1)ds\right.$$

$$-\frac{1}{T}\int_0^T [f(s,\gamma(s,\alpha,\beta,\mu),x^0(t,\gamma(s,\alpha,\beta,\mu),{x_0}^2)ds]ds$$

$$\leq\frac{K}{T}\int_0^T M_\gamma\ |x^0(s,\gamma(s,\alpha,\beta,\mu),{x_0}^1)-x^0(s,\gamma(s,\alpha,\beta,\mu),{x_0}^2)|ds$$

$$\leq M_\gamma K\frac{T}{2}.\frac{2}{T}|x^0(t,\gamma(t,\alpha,\beta,\mu),{x_0}^1)-x^0(t,\gamma(t,\alpha,\beta,\mu),{x_0}^2)|$$

$$=\frac{2}{T}\Lambda|x^0(t,\gamma(t,\alpha,\beta,\mu),{x_0}^1)-x^0(t,\gamma(t,\alpha,\beta,\mu),{x_0}^2)|$$

and hence

$$|\Delta(0,\gamma(t,\alpha,\beta,\mu),{x_0}^1)-\Delta(0,\gamma(t,\alpha,\beta,\mu),{x_0}^2)|$$

$$\leq\frac{2}{T}\ \Lambda\ |x^0(t,\gamma(t,\alpha,\beta,\mu),{x_0}^1)-x^0(t,\gamma(t,\alpha,\beta,\mu),{x_0}^2)|M_\gamma\ \cdots(3.10)$$

where ${x_0}^1(t,\gamma(t,\alpha,\beta,\mu,x_0)$ and ${x_0}^2(t,\gamma(t,\alpha,\beta,\mu,x_0)$ are the solution of the integral equation:

$$x(t,\gamma(t,\alpha,\beta,\mu),{x_0}^k)={x_0}^k+\int_0^t[f(s,\gamma(s,\alpha,\beta,\mu),x(s,\gamma(s,\alpha,\beta,\mu),{x_0}^k)$$

$$-\frac{1}{T}\int_0^T(f(s,\gamma(s,\alpha,\beta,\mu),x(s,\gamma(s,\alpha,\beta,\mu),{x_0}^k)ds]ds\ \cdots(3.11)$$

with

${x_0}^k(t,\gamma(t,\alpha,\beta,\mu),x_0)={x_0}^k,\qquad k=1,2.$

From (3.11), we have:

$$|x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^1) - x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^2)| \leq |x_0{}^1 - x_0{}^2|$$

$$+K[\left(1-\frac{t}{T}\right)\int_0^t M_\gamma\ |x^0(s,\gamma(s,\alpha,\beta,\mu),x_0{}^1) - x^0(s,\gamma(s,\alpha,\beta,\mu),x_0{}^2)|ds$$

$$+\frac{t}{T}\int_t^T M_\gamma\,|x^0(s,\gamma(s,\alpha,\beta,\mu),x_0{}^1) - x^0(s,\gamma(s,\alpha,\beta,\mu),x_0{}^2)|ds]$$

$$\leq |x_0{}^1 - x_0{}^2| + M_\gamma K\frac{T}{2}|x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^1) - x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^2)|\alpha(t)$$

$$\leq |x_0{}^1 - x_0{}^2| + \Lambda|x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^1) - x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^2)|$$

Thus

$$|x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^1) - x^0(t,\gamma(t,\alpha,\beta,\mu),x_0{}^2)| \leq (E-\Lambda)^{-1}|x_0{}^1 - x_0{}^2| \cdots (3.12)$$

Using the inequality (3.12) in (3.10), we get (3.9).

Remark 4.1[8]. The theorem 4.1 ensure the stability solution of the system (1.1) that is when there is a slight change happen in the point x_0,then a slight change will happen in the function $\Delta(t,\gamma(t,\alpha,\beta,\mu),x_0)$.

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