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# Mixed Formulation of Spline Finite Strip Method Using Equally Spaced $B_2$-Spline Series

**Abdarrhim M. Ahmed[1] , Prof. Saleh Y. Barony[2]**

[1] Basic Science Department , College of Electronic Technology
Bani-walid , ENG.Abdarrhim@Gmail.com , Libya

[2] Civil Eng. Department , School of Engng. , Academy of graduate studies
Tripoli , sybarony@yahoo.com , Libya

**Abstract**

A finite strip for analysis of plate bending based on a mixed variational formulation principle using adapted equally spaced quadratic B-Spline series as interpolation function is proposed. The advantage of this process over the draw backs of the classical finite strip method and better results of *primary as well as secondary variables* are obtained with the same order of accuracy at once rather than post calculations in conventional spline finite strip method.

**Keywords:** Finite Strip Method , Mixed Formulation, Plate Bending , $B_2$-Spline Interpolation.

## 1. Introduction

Definitely displacement-based numerical methods has wide applications for the analysis of engineering mechanics field. The most well known stiffness-based (***C****onventional* ) **F**inite **E**lement **M**ethod (CFEM) may be considered as the major one for solving various engineering problems , which utilizes higher order polynomials as interpolation functions due to continuity requirements. Such process leads to computational difficulties rather than the case of only solution variables are the displacements as the primary variables. So the displacement boundary conditions which referred to as essential (*geometric*) boundary conditions are satisfied exactly ,but the post calculated stresses named secondary variables show that the force boundary conditions which referred to as Natural (*dynamic*) boundary conditions are satisfied only in the limit as the number of elements or strips increases. [1,2,3,4]

A more general and flexible formulation is obtained by using variational principles that can be regarded as extension of the principle of stationarity of total potential energy. This principle uses the *primary variables* as well as secondary variables in the conventional formulation as dependent variables and thus its objective is the determination of *secondary variables* with the same order of accuracy directly rather than from post computations [5,6]. The same extension is applicable to the finite strip method [7].

## 2. Variational Formulation

Seeking approximate solutions to satisfy a given differential equation in a weighted integral statement such as $\int_0^l \varphi(x)\, R\, dx = 0$ where $R$ is called residual or error in the differential equation and $\varphi(x)$ is called a weight function which introduced to provide means for obtaining as many independent relations as there are unknown coefficients in the approximation is known as the weighted-residual method. To arrive to these relations (*i.e. algebraic equations*) the next three steps are followed [8]:

1. Multiply the governing differential equation with a weight function and recast it in weighted integral form over its domain
2. Trade the differentiation from its dependent variable to the weight function using integration by parts or the Green-Gauss theorem, this referred to as weaken the continuity hence the name of the procedure is known as the weak form of the governing differential equation
3. Identify the primary and secondary variables of the variational (or weak ) form, by examining the boundary term appearing in the weak form set , such that the coefficient of the weight function is called secondary variable and its specification constitute the *natural boundary conditions and the dependent unknown in the same form as the weight function is termed the primary variable and its* specification constitute the *essential boundary conditions.*

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## 3. Mixed Formulation of Plate Bending Spline Finite Strip

As mixed method for plate bending problems was suggested by Herrmann [9] has been further developed in sequel however more attention has been focused on the mixed finite element method for plate and shell [10,11] , only a few researches are concerned with mixed strip method [7,12]. Although the Hellinger-Reissner functional can be used directly, alternatively , this article present a mixed spline strip using $B_2$-Spline as interpolation function for all field variables in plate bending problem through weak form formulation.

### 3.1. Basic Relations and Governing Differential Equation

The following relationship relating bending and twisting moments $m_x$ , $m_y$ , $m_{xy}$ with either deflection $w$ or applied load $q$ are deduced from orthotropic plate theory where $D_{ij}$ are the bending stiffness's of the plate:

$$m_x = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right)$$

$$m_y = -\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{xy} = -2D_{33}\frac{\partial^2 w}{\partial x \partial y}$$

, solve for $\frac{\partial^2 w}{\partial x^2}$ , $\frac{\partial^2 w}{\partial y^2}$ & $\frac{\partial^2 w}{\partial x \partial y}$ yields:

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= -(\widetilde{D}_{22}\, m_x - \widetilde{D}_{12} m_y) \\ \frac{\partial^2 w}{\partial y^2} &= -(\widetilde{D}_{12} m_x - \widetilde{D}_{11} m_y) \\ 2\frac{\partial^2 w}{\partial x \partial y} &= -\widetilde{D}_{33} m_{xy} \end{aligned}\right\} \text{------(1)}$$

*where*

$$\widetilde{D}_{11} = \frac{D_{11}}{D_{11}D_{22} - D_{12}^2} \quad , \quad \widetilde{D}_{12} = \frac{D_{12}}{D_{11}D_{22} - D_{12}^2}$$

$$\widetilde{D}_{22} = \frac{D_{22}}{D_{11}D_{22} - D_{12}^2} \quad , \quad \widetilde{D}_{33} = (D_{33})^{-1} \quad \text{and}$$

$$-\left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}\right) = q \quad \text{-------(2)}$$

### 3.2. Weak Form

Applying weak form procedure for the set of equations (Eq.1 & Eq.2) , where variation symbol $\delta$ is omitted here yields the following:

$$0 = \int_R \left\{\frac{\partial w}{\partial x}\frac{\partial m_x}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial m_{xy}}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial m_y}{\partial y} + \right.$$
$$\left. + \frac{\partial w}{\partial y}\frac{\partial m_{xy}}{\partial x} - qw\right\} dR - \oint_C \left\{w\frac{\partial m_x}{\partial x}\boldsymbol{n}_x + \right.$$
$$\left. + w\frac{\partial m_{xy}}{\partial y}\boldsymbol{n}_x + w\frac{\partial m_y}{\partial y}\boldsymbol{n}_y + w\frac{\partial m_{xy}}{\partial x}\boldsymbol{n}_y\right\} ds$$

$$0 = \int_R \left\{\frac{\partial w}{\partial x}\frac{\partial m_x}{\partial x} - \widetilde{D}_{22} m_x m_x + \widetilde{D}_{12} m_x m_y\right\} dR - \oint_C \left\{m_x\frac{\partial w}{\partial x}\boldsymbol{n}_x\right\} ds$$

$$0 = \int_R \left\{\frac{\partial w}{\partial y}\frac{\partial m_y}{\partial y} - \widetilde{D}_{11} m_y m_y + \widetilde{D}_{12} m_y m_x\right\} dR - \oint_C \left\{m_y\frac{\partial w}{\partial y}\boldsymbol{n}_y\right\} ds$$

$$0 = \int_R \left\{\frac{\partial w}{\partial x}\frac{\partial m_{xy}}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial m_{xy}}{\partial x} + \widetilde{D}_{33} m_{xy} m_{xy}\right\} dR - \oint_C \left\{m_{xy}\frac{\partial w}{\partial y}\boldsymbol{n}_x + m_{xy}\frac{\partial w}{\partial x}\boldsymbol{n}_y\right\} ds$$

*where $\boldsymbol{n}_x$ & $\boldsymbol{n}_y$ are the direction cosines of the unit normal to the boundary.*

### 3.3. Interpolation Function

Utilizing an appropriate interpolation functions for field variables $\{m_x , m_y , m_{xy} , w\}$, the following augmented matrix can be obtained from the weak form set as:

$$[K_s] = \begin{bmatrix} [K_{11}] & [K_{12}] & [K_{13}] & [K_{14}] \\ [K_{21}] & [K_{22}] & [K_{23}] & [K_{24}] \\ [K_{31}] & [K_{32}] & [K_{33}] & [K_{34}] \\ [K_{41}] & [K_{42}] & [K_{43}] & [K_{44}] \end{bmatrix}$$

*where if $[B]$ is the strain matrix of the field variable then:*

$$[K_{11}] = -\widetilde{D}_{22}\int_0^a\int_0^b [B_{mx}]^T [B_{mx}]\, dx\, dy$$

$$[K_{12}] = -\widetilde{D}_{12}\int_0^a\int_0^b [B_{mx}]^T [B_{my}]\, dx\, dy$$

$$[K_{14}] = \int_0^a\int_o^b \frac{\partial}{\partial x}[B_{mx}]^T \frac{\partial}{\partial x}[B_w]\, dx\, dy$$

$$[K_{21}] = -\widetilde{D}_{11}\int_0^a\int_0^b [B_{my}]^T [B_{mx}]\, dx\, dy$$

$$[K_{22}] = -\widetilde{D}_{12}\int_0^a\int_0^b [B_{my}]^T [B_{my}]\, dx\, dy$$

$$[K_{24}] = \int_0^a\int_o^b \frac{\partial}{\partial y}[B_{my}]^T \frac{\partial}{\partial y}[B_w]\, dx\, dy$$

$$[K_{23}] = \widetilde{D}_{33}\int_0^a\int_0^b [B_{mxy}]^T [B_{mxy}]\, dx\, dy$$

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$$[Q] = [K_{34}] = \int_0^a \int_0^b \left\{ \frac{\partial}{\partial y} [B_{mxy}]^T \frac{\partial}{\partial x} [B_w] + \frac{\partial}{\partial x} [B_{mxy}]^T \frac{\partial}{\partial y} [B_w] \right\} dx\, dy - [\widetilde{K}_{34}]$$

$$[\widetilde{K}_{34}] = = \int_0^b \left\{ [B_{mxy}]^T \frac{\partial}{\partial x} [B_w] \Big|_{y=a} - [B_{mxy}]^T \frac{\partial}{\partial x} [B_w] \Big|_{y=0} \right\} dx + \int_0^a \left\{ [B_{mxy}]^T \frac{\partial}{\partial y} [B_w] \Big|_{x=b} - [B_{mxy}]^T \frac{\partial}{\partial y} [B_w] \Big|_{x=0} \right\} dy$$

$$[K_{41}] = [K_{14}]^T \;,\; [K_{42}] = [K_{24}]^T,\; [Q^*] = [K_{34}]^T$$

$$[K_{13}] = [K_{23}] = [K_{31}] = [K_{32}] = [K_{44}] = [0]$$

For isotropic plate: $\widetilde{D}_{22} = \widetilde{D}_{11} = \eta = 12/E\tau^3$ $\widetilde{D}_{12} = \nu\eta$ , $\widetilde{D}_{33} = 24(1+\nu)/E\tau^3 = 2\eta(1+\nu)$
where $E$ is the modulus of elasticity , $\tau$ plate thickness and $\nu$ is a passion ratio.

### 3.4. $B_2$-Spline Function

As a spline become a mathematical tool after the seminal work of Schoenberg [2 ,13] . A variety studies and applications of spline functions has been developed in sequel such as: "Calculating with B-splines" in1972 [14], "Using of B-splines for computer-aided design" [15] and the very important paper [16] presented by **James M. McKee** in 1977 , on use of the linear combination of the basis functions of typical $B_3$-splines (*i.e. spline series , similar to the one shown in figure* 3) as interpolating and approximating function which converges to the original one as the number of knots increases if the knot points (nodal values) are substituted for the coefficients of that combination.

**Consequently** Y.K. Cheung , Fan and Wu in 1982 [17] , also they made use of the $B_3$-*spline series* as interpolant instead of trigonometric series in the longitudinal direction of the strip to overcome the difficulties arising in the classical finite strip method which provide the well known today as the *Spline Finite Strip Method*.

An examination of the above weak form shows that the $C^0$ continuity is required in this mixed formulation, therefore the lower order $B_2$-spline is used in this present work as the series part of interpolation function for each of the field variables , but instead of the well known standard quadratic $B_2$-spline , a little modified one will be used here in order to obtain the equally spaced $B_2$-Spline Series which is in both conventional and normalized forms as:

$$\phi(y) = \begin{cases} \frac{1}{3h^2} y^2 & 0 \le y \le h \\ \frac{-1}{3h^2}(y^2 - 4yh + 2h^2 & h \le y \le 3h \\ \frac{1}{3h^2}(y^2 - 8yh + 16h^3) & 3h \le y \le 4h \end{cases}$$

$$\phi(t) = \begin{cases} \frac{1}{12} t^2 + \frac{1}{6} t + \frac{1}{12} & -1 \le t \le 1 \\ \frac{1}{12} - \frac{1}{3} t^2 & -1 \le t \le 1 \\ \frac{1}{12} - \frac{1}{6} t + \frac{1}{12} t^2 & -1 \le t \le 1 \end{cases}$$

$$\phi'(y) = \begin{cases} \frac{2}{3h^2} y & 0 \le y \le h \\ \frac{-1}{3h^2}(2y - 4h) & h \le y \le 3h \\ \frac{1}{3h^2}(2y - 8h) & 3h \le y \le 4h \end{cases}$$

$$\phi'(t) = \begin{cases} \frac{1}{3h}(1+t) & -1 \le t \le 1 \\ -\frac{2}{3h} t & -1 \le t \le 1 \\ \frac{1}{3h}(t-1) & -1 \le t \le 1 \end{cases}$$

*where t is a parameter.*

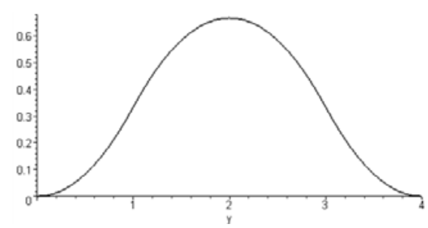

Fig. 1 $B_2$-Spline function $\phi(y)$

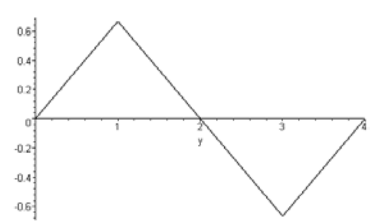

Fig. 2 $\phi'(y)$ ; 1st derivative of $\phi(y)$

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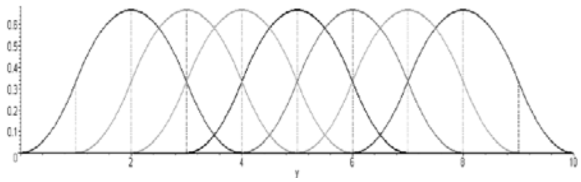

Fig. 3 Equally Spaced $B_2$-Spline Series

*Note:*
*for convenience in programming, the middle arc of $B_2$-spline is treated as two segments*

### 3.5. Mixed Spline Strip Model

Utilizing the equally spaced $B_2$-Spline in the longitudinal direction while the polynomial of first degree in the transverse direction for the field variables $\{m_x, m_y, m_{xy}, w\}$, is sufficient to formulate a low order plate bending strip. As an example, the field variable $m_x$ may be represented as follows:

$$m_x = \sum_{k=-1}^{m+1} (a_0 + a_1 x)_k \ \alpha_{mx_k} \phi_k^{m_x}(y)$$

$$= \begin{bmatrix} 1-\frac{x}{b} & \frac{x}{b} \end{bmatrix} \begin{bmatrix} \{\sum_{k=-1}^{m+1} \alpha_{mx_k} \phi_k^{m_x}(y)\}_i \\ \{\sum_{k=-1}^{m+1} \alpha_{mx_k} \phi_k^{m_x}(y)\}_j \end{bmatrix}$$

$$= [N] \begin{bmatrix} [\Phi_{m_x}]_i & [0] \\ [0] & [\Phi_{m_x}]_j \end{bmatrix} \begin{bmatrix} [\delta_{m_x}]_i \\ [\delta_{m_x}]_j \end{bmatrix}$$

$$= [N][\Psi_{m_x}][\Delta_{m_x}] \ , \text{ therefore}$$

$$m_x = [N][\Psi_{m_x}][\Delta_{m_x}] = [B_{m_x}][\Delta_{m_x}]$$

$$m_y = [N][\Psi_{m_y}][\Delta_{m_y}] = [B_{m_y}][\Delta_{m_y}]$$

$$m_{xy} = [N][\Psi_{m_{xy}}][\Delta_{m_{xy}}] = [B_{m_{xy}}][\Delta_{m_{xy}}]$$

$$w = [N][\Psi_w][\Delta_w] = [B_w][\Delta_w]$$

where for any nodal line *i* or *j* in the problem :
$[\Phi_p] = [\phi_{-1}^p \ \ \phi_0^p \ \ \phi_1^p \ \ \cdots \ \ \phi_m^p \ \ \phi_{m+1}^p]$ ,
are local $B_2$-splines to interpolate the field variable $p$
$[\delta_p] = [\alpha_{p_{-1}} \alpha_{p_0} \ \alpha_{p_1} \ \cdots \ \alpha_{p_m} \ \alpha_{p_{m+1}}]^T$ ,
are nodal parameters vector of the field variable $p$
$[N] = [n_1 \quad n_2]$ are the shape functions along any section *m* which are the same for all sections.

Substitution of these approximation functions into sub matrices of augmented matrix $[K_s]$ and performing the necessary multiplications and integrations yields the final explicit form of $[K_s]$, for example:

$$[K_{11}] = -\widetilde{D}_{22} \int_0^a \int_0^b [B_{mx}]^T [B_{mx}] \, dx \, dy$$

$$= -\eta \int_0^a \int_0^b [\Psi_{m_x}]^T [N]^T [N] [\Psi_{m_x}] \, dx \, dy [\Delta_{m_x}]$$

$$[K_{11}] = \int_0^a \int_0^b [\Psi_{m_x}]^T [A] [\Psi_{m_x}] \, dx \, dy [\Delta_{m_x}]$$

where [A] and the other its similarities as it can be observed from weak form with the following notations:

$$[A] = -\eta \int_0^a [N]^T [N] \, dx \ , [B] = \nu\eta \int_0^a [N]^T [N] \, dx$$

$$[C] = \int_0^a [N']^T [N'] \, dx \ , [D] = \int_0^a [N]^T [N] \, dx$$

$$[E] = 2\eta(1+\nu) \int_0^a [N]^T [N] \, dx$$

$$[F] = \int_0^a [N]^T [N'] \, dx$$

$$[V] = [N][N']\big|_{x=b} - [N][N']\big|_{x=0}$$

Direct evaluations reveals if $[M] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ then :

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = -\tfrac{1}{6}\eta b \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = -\tfrac{1}{6}\eta b [M]$$

$[B] = \nu\eta b/6[M]$ , $[D] = b/6[M]$
$[E] = (1+\nu)\eta b/3[M]$
$[C] = 1/b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ , $[F] = 1/2 \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$
$[G] = [F]^T$ , $[V] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \ [K_{11}] = -\eta \int_0^a \int_0^b \begin{bmatrix} a_{11}[\Phi_{m_x}]^T[\Phi_{m_x}] & a_{12}[\Phi_{m_x}]^T[\Phi_{m_x}] \\ a_{21}[\Phi_{m_x}]^T[\Phi_{m_x}] & a_{22}[\Phi_{m_x}]^T[\Phi_{m_x}] \end{bmatrix} dxdy$$

introducing the following notation for each type of integrations involved in $[K_s]$:

$$\mathbf{I_1} = \int_0^a [\Phi_{mx}]^T [\Phi_{mx}] \, dy \ , \mathbf{I_2} = \int_0^a [\Phi_{mx}]^T [\Phi_{my}] \, dy$$

$$\mathbf{I_3} = \int_0^a [\Phi_{mx}]^T [\Phi_w] \, dy \ , \mathbf{I_4} = \int_0^a [\Phi_{my}]^T [\Phi_{my}] \, dy$$

$$\mathbf{I_5} = \int_0^a [\Phi'_{my}]^T [\Phi'_w] \, dy \ , \mathbf{I_6} = \int_0^a [\Phi_{mxy}]^T [\Phi_{mxy}] \, dy$$

$$\mathbf{I_7} = \int_0^a [\Phi'_{mxy}]^T [\Phi_w] \, dy \ , \mathbf{I_8} = \int_0^a [\Phi_{mxy}]^T [\Phi'_w] \, dy$$

*where* $\Phi'_p \equiv \frac{\partial \Phi_p}{\partial y}$

$$\mathbf{IB} = [\Phi_{mxy}]^T [\Phi_w]\big|_{y=a} - [\Phi_{mxy}]^T [\Phi_w]\big|_{y=0}$$

Although the strip has eight degrees of freedom (DOF) four per each nodal line, that are $\alpha_{mx_i}, \alpha_{my_i}, \alpha_{mxy_i}, \alpha_{w_i}$ and $\alpha_{mx_j}, \alpha_{my_j}, \alpha_{mxy_j}, \alpha_{w_j}$, each elemental involving eight knots four per each nodal line [18] which in turns provide thirty two DOF.

Noting that the first element in each sub-matrix which is corresponding to the integration of first coupled terms of the spline series where in case of sub-matrix $[K_{11}]$ is $I_1^{-1,-1} = \int_{y_1}^{y_2} \phi_{-1}^{T_{mx}} \phi_{-1}^{mx} dy$, consequently in general any element in each sub-matrix of $[K_s]$ corresponding to the coupled terms integration $I_\chi^{i,j} = \int_{y_1}^{y_2} \phi_i^{T_p} \phi_j^{p^*} dy$, where $\chi = 1, 2, \dots, 8$ , $i = j = -1, \dots, m+1$ and the field variables $p$ & $p^* \in \{m_x, m_y, m_{xy}, w\}$ provides a **General Augmented Basic-Matrix** $[K_b^{i,j}]_{8\times8}$ of this formulation which in turn yields the **Elemental Augmented Matrix** $[\mathrm{K_e}]_{32\times32}$ having the following rearrangement:

$$[\mathrm{K_e}]_{32\times32} =$$

$$For\ \mathcal{L} = 1\ To\ 4 :\quad For\ \ell = 1\ To\ 4$$

$$[K_b^{i,j}]_{8\times8}^{\mathcal{L},\ell} =$$

$$\begin{bmatrix}
a_{11}\mathrm{I}_1^{i,j} & b_{11}\mathrm{I}_2^{i,j} & 0 & c_{11}\mathrm{I}_3^{i,j} & a_{12}\mathrm{I}_1^{i,j} & b_{12}\mathrm{I}_2^{i,j} & 0 & c_{12}\mathrm{I}_3^{i,j} \\
b_{11}\mathrm{I}_2^{i,j} & a_{11}\mathrm{I}_4^{i,j} & 0 & d_{11}\mathrm{I}_5^{i,j} & b_{12}\mathrm{I}_2^{i,j} & a_{12}\mathrm{I}_2^{i,j} & 0 & d_{12}\mathrm{I}_5^{i,j} \\
0 & 0 & e_{11}\mathrm{I}_6^{i,j} & q_{11} & 0 & 0 & e_{12}\mathrm{I}_6^{i,j} & q_{12} \\
c_{11}\mathrm{I}_3^{i,j} & d_{11}\mathrm{I}_5^{i,j} & q_{11}^* & 0 & c_{12}\mathrm{I}_3^{i,j} & d_{12}\mathrm{I}_5^{i,j} & q_{12}^* & 0 \\
a_{21}\mathrm{I}_1^{i,j} & b_{21}\mathrm{I}_2^{i,j} & 0 & c_{21}\mathrm{I}_3^{i,j} & a_{22}\mathrm{I}_1^{i,j} & b_{22}\mathrm{I}_2^{i,j} & 0 & c_{22}\mathrm{I}_3^{i,j} \\
b_{21}\mathrm{I}_2^{i,j} & a_{21}\mathrm{I}_4^{i,j} & 0 & d_{21}\mathrm{I}_5^{i,j} & b_{22}\mathrm{I}_2^{i,j} & a_{22}\mathrm{I}_4^{i,j} & 0 & d_{22}\mathrm{I}_5^{i,j} \\
0 & 0 & e_{21}\mathrm{I}_6^{i,j} & q_{21} & 0 & 0 & e_{22}\mathrm{I}_6^{i,j} & q_{22} \\
c_{21}\mathrm{I}_3^{i,j} & d_{21}\mathrm{I}_5^{i,j} & q_{21}^* & 0 & c_{22}\mathrm{I}_3^{i,j} & d_{22}\mathrm{I}_5^{i,j} & q_{22}^* & 0
\end{bmatrix}_{8\times8}$$

$$Next\ \mathcal{L}\ :\quad Next\ \ell$$

Corresponding to the nodal parameters vector:

$$[\alpha_{mx_i}\quad \alpha_{my_i}\quad \alpha_{mxy_i}\quad w_i\quad \alpha_{mx_j}\quad \alpha_{my_j}\quad \alpha_{mxy_j}\quad w_j]^T$$

## 4. Illustrative Examples

Instead of low order strip , for better accuracy the higher order one was implemented in computer program and the results are presented herein. Using the above formulation as well as the derived $B_2$-spline series which leads to the high order explicit augmented matrix as given by [19], the following numerical examples are presented , where due to symmetry , only half of the square plate with a uniform distributed load as shown in figure (4) , was analyzed under the following conditions:

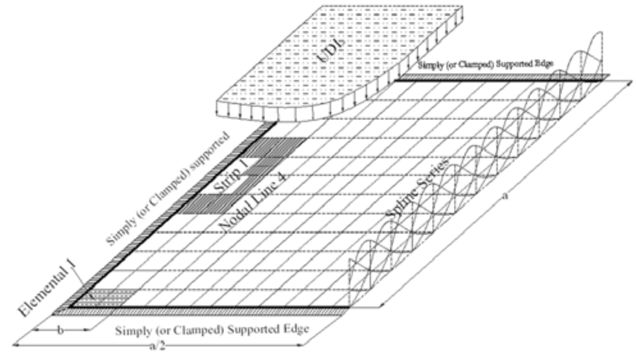


Fig. 4 Half of square plate divided into high order strips

### 4.1. Simply supported case

***a.*** Prescribed boundary conditions which should be satisfied by spline part priori:

$$\Phi_{m_x}(0) = \Phi_{m_y}(0) = \Phi_w(0) = 0$$
$$\Phi_{m_x}(a) = \Phi_{m_y}(a) = \Phi_w(a) = 0$$

***b.*** Prescribed boundary conditions which should be imposed in augmented matrix:

$$\alpha_{m_x}(x) = \alpha_{m_y}(x) = \alpha_w(x) = 0 \Big|_{x=0}$$
$$\alpha_{m_{xy}}(x) = 0 \Big|_{x=a/2} \quad , \quad \alpha_{m_{xy}}(y) = 0 \Big|_{y=a/2}$$

***c.*** For satisfactory results the following nodal parameters should be eliminated at sections:

$$\alpha_{m_x} = \alpha_{m_y} = \alpha_w = 0 \Big|_{section\ =-1,0}$$
$$\alpha_{m_x} = \alpha_{m_y} = \alpha_w = 0 \Big|_{section\ =m,m+1}$$

The computational results as program output are given in Tables [1&2].

Table [1]

Spline Finite Strip Method
Mixed Formulation Using B2-Splines

The Case is all Simply    With Uniformly Distributed Loading

NO. OF STRIPS= 2    NO. OF SECTIONS= 4    Total DOF= 140

The analytical solutions are:
wc=0.004062    mxc=0.0479    myc=0.0479    mxye=0.0324

| Program output | W | mx | my | mxy |
|---|---|---|---|---|
| at the center of plate —— | 0.004068 | 0.047979 | 0.048057 | 0.000000 |
| at the center of edge y=0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| at the center of edge y=a | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| at the center of edge x=0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| at the corner (x=0 , y=0) | 0.000000 | 0.000000 | 0.000000 | -0.035465 |

Table [2]

Spline Finite Strip Method
Mixed Formulation Using B2-Splines

The Case is all Simply     With Uniformly Distributed Loading

NO. OF STRIPS= 4     NO. OF SECTIONS= 8     Total DOF= 396

The analytical solutions are:
wc=0.004062   mxc=0.0479   myc=0.0479   mxye=0.0324

| Program output | W | mx | my | mxy |
|---|---|---|---|---|
| at the center of plate —— | 0.004062 | 0.047892 | 0.047896 | 0.000000 |
| at the center of edge y=0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| at the center of edge y=a | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| at the center of edge x=0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| at the corner (x=0 , y=0) | 0.000000 | 0.000000 | 0.000000 | -0.033460 |

## 4.2. Clamped supported case

***a.*** Prescribed boundary conditions which should be satisfied by spline part priori:

$$\Phi_{m_x}(0) = \Phi_{m_{xy}}(0) = \Phi_w(0) = 0$$

$$\Phi_{m_x}(a) = \Phi_{m_{xy}}(a) = \Phi_w(a) = 0$$

***b.*** Prescribed boundary conditions which should be imposed in augmented matrix:

$$\alpha_{m_y}(x) = \alpha_{m_{xy}}(x) = \alpha_w(x) = 0 \Big|_{x=0}$$

$$\alpha_{m_{xy}}(x) = 0 \Big|_{x=a/2} \quad , \quad \alpha_{m_{xy}}(y) = 0 \Big|_{y=a/2}$$

***c.*** For satisfactory results the following nodal parameters should be eliminated at sections:

$$\alpha_{m_x} = \alpha_{m_{xy}} = \alpha_w = 0 \Big|_{section=-1,0}$$

$$\alpha_{m_x} = \alpha_{m_{xy}} = \alpha_w = 0 \Big|_{section=m,m+1}$$

The computational results as program output are given in Tables [3&4]

Table [3]

Spline Finite Strip Method
Mixed Formulation Using B2-Splines
The case is all Fixed     With Uniformly Distributed Loading

NO. OF STRIPS= 2   NO. OF SECTIONS= 4   Total DOF= 140

The analytical solutions are:
wc=0.00126   mxc=0.0231   myc=0.0231   my or mx (mid edges)= -0.0513

| Program output | W | mx | my | mxy |
|---|---|---|---|---|
| at the center of plate —— | 0.001322 | 0.024939 | 0.024136 | 0.000000 |
| at the center of edge y=0 | 0.000000 | 0.000000 | -0.051780 | 0.000000 |
| at the center of edge y=a | 0.000000 | 0.000000 | -0.051780 | 0.000000 |
| at the center of edge x=0 | 0.000000 | -0.052444 | 0.000000 | 0.000000 |
| at the corner (x=0 , y=0) | 0.000000 | 0.000000 | 0.000000 | 0.000000 |

Table [4]

Spline Finite Strip Method
Mixed Formulation Using B2-Splines
The case is all Fixed     With Uniformly Distributed Loading

NO. OF STRIPS= 6   NO. OF SECTIONS= 12   Total DOF= 780

The analytical solutions are:
wc=0.00126   mxc=0.0231   myc=0.0231   my or mx (mid edges)= -0.0513

| Program output | W | mx | my | mxy |
|---|---|---|---|---|
| at the center of plate —— | 0.001277 | 0.023066 | 0.023002 | 0.000000 |
| at the center of edge y=0 | 0.000000 | 0.000000 | -0.051046 | 0.000000 |
| at the center of edge y=a | 0.000000 | 0.000000 | -0.051046 | 0.000000 |
| at the center of edge x=0 | 0.000000 | -0.051182 | 0.000000 | 0.000000 |
| at the corner (x=0 , y=0) | 0.000000 | 0.000000 | 0.000000 | 0.000000 |

## 5. Conclusion

With a relatively small number of strips , a very good accordance results if it`s compared with the analytical solutions given in [20], can be observed as shown in Tables [1 , 2 , 3 & 4].

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First Author BSc.(1984),MSc.(2007),Corps of projects,Higher institute of electronics,college of electronics and technology at Bniwalid.national conference of building materials and structural enineering ,PhD. research grant,5 papers,finite element and finite strip methods.
Seconf Author BSc.(1966),MSc.(1969),PhD(1977),prof. of civil eng.,Dean of college of eng.at Libyan academy of graduate studies,sapatecal leave for one year(1984),head of deffirent national committees.