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Properties of configurations of four color theorem

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Abstract: this paper studies the concept of configurations and triangulations of Four color theorem..

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Introduction *minimal counterexample* is a plane graph G which is not 4-colorable such that every graph G' with $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ is four-colorable. From this definition, we may show that every minimal counterexample is a triangulation

Configurations-1

We will consider the four-color problem on the 2-sphere and on the plane; as mentioned above these are equivalent conditions. When considering planar graphs there will be one region designated as the *infinite region* , all the others are finite. In figures (i.e., plane drawings) representing planar drawings the outside is considered infinite.

2.1 Triangulations

A region (or face) of a plane graph is called a *triangle* if it is incident with exactly 3 edges. A connected, loopless plane graph on the sphere is called a *triangulation* if every region is a triangle. In particular, no circuit of length 2 bounds a region of a triangulation. A *near-triangulation* is a non-null connected planar graph G in which every finite region is a triangle.

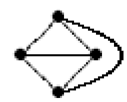

Figure 4.1: A triangulation. Note this graph is 4-colorable but not 3-colorable.

2.2Theorem *To prove the 4CT for graphs on the 2-sphere, it is sufficient to prove the 4CT for triangulations.*

Proof. Let G be a graph on the 2-sphere S^2. Let $F(G)$ be the faces of G. Then every $f \in F(G)$ is a polygon projected onto the sphere. Say f is an n-gon, $n > 3$, with vertices $v_1, ..., v_n$, add vertex v_f to the face of f and draw edges $v_i v_f$ for $i = 1, ..., n$. Do this for each n-gon $f \in F(G)$ such that $n > 3$ to obtain triangulation G'. Suppose there exists a four-coloring(v) for $c' : V(G') \rightarrow \{1, 2, 3, 4\}$, then c' restricts to a four-coloring on G.

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2.3Corollary *To prove the 4CT for graphs on the plane, it is sufficient to prove the 4CT for near-triangulations(1)*

2.4Minimal Counterexamples

Suppose there is a map which is not four-colorable. Then there is such a map with the fewest number of countries. Such a map would be a minimal counterexample; any map with fewer countries can be colored with four colors. We wish to show that no minimal counterexamples exist. We define this concept in terms of vertex-colorings on planar graphs.

A *minimal counterexample* is a plane graph G which is not 4-colorable such that every graph G' with $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ is four-colorable. From this definition, we may show that every minimal counterexample is a triangulation.

2.5Theorem *Every minimal counterexample is a triangulation.*

Proof. Let G be a minimal counterexample. If there is a region of G bounded by a circuit of length 2, say e_1e_2 where $e_1, e_2 \in E(G)$ are incident with vertices v and w then replace e_1 and e_2 with a single edge vw to obtain graph G'. Clearly, this will not effect the vertex coloring of G but $|V(G')| + |E(G')| < |V(G)| + |E(G)|$, a contradiction. Thus G cannot have a region bounded by a circuit of length 2.

Now assume G has a region bounded by a circuit of length $n \geq 4$. Let v_i, i

=1, ..., n be the n vertices on the boundary of the region. Without loss of generality, we may choose 2 of these vertices, say v_1, v_2 which are not adjacent. Draw edge v_1v_2 and contract it to a vertex v, call the new graph G'. Graph G' must be 4-colorable with coloring $c' : V(G') \to \{1, 2, 3, 4\}$. Thus, we may define a 4-coloring, c, on G by: $c(u) = c'(u)\ \forall\ u \in V(G) \cap V(G')$ and $c(v_1) = c(v_2) = c'(v)$, a contradiction.

Birkhoff(3) proved additional properties of minimal counterexamples in 1913 (Birkhoff, 1913). He examined a particular type of circuit which he called a "ring" [1]. Birkhoff's notion of a ring is a circuit with vertices $\{v_1, ..., v_n\}$ such that v_i is a neighbor of v_j if and only if $|i - j| = 1$ or $n - 1$; the ring-size is n. Note that we will define "ring" and "ring-size" differently (in alignment with Robertson et al) in the previous section. By using a Kempe Chain argument he showed that a minimal counterexample can have neither:

1. A ring of size 4, nor
2. A ring of size 5 which contains more than one vertex.

We prove Theorem 4.15.1 below which implies (1) and(2).

For a minimal counterexample G, there is an equivalent way to describe properties (1) and (2) from above. Robertson et al define a *short circuit* of a triangulation as a circuit C with $|E(C)| \leq 5$, so that for both open discs Δ on the sphere bounded by $C, \Delta \cap V(G) \neq \emptyset$, and $|\Delta \cap V(G)| \geq 2$ if $|E(C)| = 5$. In other words, a short circuit with 4 edges has vertices in both its interior and exterior. If it has exactly 5 edges then there are at least 2 vertices in the interior and exterior. We say G is *internally six-connected* if it has no short circuit. The above discussion gives us the following theorem.

2.6Theorem. *Every minimal counterexample is an internally 6-connected triangulation.*

Thus, there are relatively few types of circuits of short length in a minimal counterexample G. We have shown in Lemma 2.1 that there are no circuits of length 2.

There are plenty of circuits of length 3, namely triangles. Examples of permissible and non-permissible circuits (along with interior edges) of length 4 and 5 are shown in Figures 4.5 and 4.6. Every circuit of size ≤ 5 consists of either the vertices of triangle(s) or are the neighbors of a single 5-vertex.

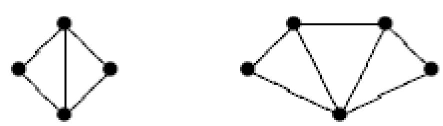

Figure 4.2: Permissible circuits of size ≤ 5 with interior edges.

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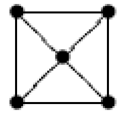

Figure 4.3: A short circuit with interior edges & vertex.

3 More on Kempe Chains

A further examination of Figure 4.6 will be an instructive demonstration of the Kempe Chain method and will give further insight into the definition of an internally 6-connected triangulation. We first make precise the discussion of Kempe chains from previous section. A *Kempe chain* in a vertex 4-colored graph G is a sequence of vertices colored with only two colors. A Kempe chain in colors b, $g \in \{1, 2, 3, 4\}$ is called a (b, g)-*chain*. A Kempe net is a component of the subgraph G_{bg}, the subgraph of G spanned by all vertices colored b or g. Hence, each pair of vertices in a Kempe net can be joined by a Kempe chain. Our previous discussion of Kempe chains in section describes the process of interchanging colors in a Kempe net. This process is called *Kempe interchange* and results in a new 4-coloring.

Figure 4.6 has a 4-vertex (a vertex of degree 4). As it appears in a triangulation, then there must be a circuit of length four about it, i.e. a short circuit. Kempe proved that no such vertex can exist in a minimal counterexample using his method of Kempe Chains, we now present the proof.

3.1Theorem *There is no vertex of degree 4 in a minimal counterexample.*

Proof. Let y be a vertex of a minimal counterexample G such that $d_G(y) = 4$. Let $x_1, ..., x_4$ be the neighbors of y in cyclic order. Remove y and its incident edges from G to obtain graph G'. Since G is a minimal counterexample there exists a vertex four-coloring for G', say $c': G' \rightarrow \{1, 2, 3, 4\}$. If only three colors are needed for the neighbors of y then there is a 4^{th} color free for y and we may extend c' to a 4-coloring of G.

Thus we assume $x_1, ..., x_4$ are colored with 4 colors. Without loss of generality, assume $c'(x_i) = i \in \{1, 2, 3, 4\}$. We consider two cases:

Case 1. If x_1 and x_3 are in different (1, 3)-nets then we may perform a Kempe interchange so that x_1 is now colored with color 3. Then y has neighbors of only colors 2, 3, 4 and c' may be extended to a 4-coloring on G by defining $c'(y) = 1$.

Case 2. If x_1 and x_3 are in same (1, 3)-net then, by our discussion in previous section, x_2 and x_4 must belong to different (2, 4)-nets and we may perform a Kempe inter-change so that x_2 is colored with color 2. Thus we may again extend c' to a 4-coloring of G.(2)

Kempe erroneously used a similar process to show that a minimal counterexample cannot have a five vertex. However, as discussed in previous section, there was a flaw in his logic as he attempted to do two Kempe interchanges simultaneously. Nonetheless, Theorem 3.1 proves that every vertex in a minimal counterexample has degree at least 5. Thus every (interior) vertex in a minimal counterexample is surrounded by a circuit of length at least 5. Thus Birkhoff showed that a minimal counterexample can have neither (1) a ring of size 4 nor (2) a ring of size 5 which contains more than one vertex as stated above. In the language of Robertson et al, it cannot have a short circuit.

Robertson et al offer another approach to proving Theorem 4.2 They are able to define a quadratic-time algorithm which constructs a 4-coloring of a triangulation G from a short circuit of G. The existence of such an algorithm and Theorem 4.2

4 Configurations -2

A *configuration* K consists of a near-triangulation $G(K)$ and a map γ_K : $V(G(K)) \to \mathbb{Z}_{\geq 0}$ with the following properties:

1. For every $v \in V(G(K))$, $G(K) \setminus \{v\}$ has at most two components, and if there are two then $\gamma_K(v) = d(v) + 2$.
2. For every $v \in V(G(K))$, if v is not incident with the infinite region, then $\gamma_K(v) = d(v)$. Otherwise, $\gamma_K(v) > d(v)$. In either case $\gamma_K(v) \geq 5$.
3. K has ring-size ≥ 2 where the *ring-size* of K is $\sum_v(\gamma_K(v) - d(v) - 1)$, summed

over all vertices v that are incident with the infinite region such that $G(K) \setminus v$ is connected.

Figure 4.4 shows examples of configurations. The vertex shapes represent the value of γ_K at that vertex. The shapes follow the conventions established by Heesch in the 1960s and used by Robertson et al in their proof. For the examples below we have two vertex shapes, " •" represents γ_K $(v) = 5$ and " ·" represents γ_K $(v) = 6$.

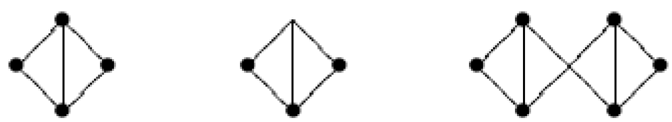

Figure 4.4: Examples of configurations.

A subgraph G' of G is *induced* if every edge of G with both ends in $V(G')$ belongs to G'. Figure 4.5 shows an example of a subgraph which is not induced. A configuration K *appears* in a graph G if $G(K)$ it is an induced subgraph of G, every finite region of $G(K)$ is a region of G, and $\gamma_K = d_G$ (v) $\forall$ $v \in V(G(K))$. This offers some more motivation for the definition of the configuration above, γ_K (v) corresponds to the degree of v in the graph in which the configuration is induced. The first and second examples in Figure 4.4 differ in how they would "appear" in a graph.

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Figure 4.5: Graph G' is a subgraph of G but it is *not an induced subgraph*.

We make some further observations about configurations. They are near-triangulations in which there are no bridges or vertices of degree 1. The outer vertices (those which are incident with the infinite region) form a closed circuit of size ≥ 4. Inner vertices always have degree ≥ 5. The ring-size condition (3) will allow us to prove the existence of free completions as defined in previous section; this condition will be elaborated upon in the proof of Proposition .(4)

4.1 Good Configurations

Robertson et al define a set of 633 configurations. Two configurations K and L are *iso-morphic* if there is a homeomorphism of the 2-sphere which maps $G(K)$ to $G(L)$ and γ_K to γ_L. A configuration isomorphic to one of these 633 configurations is called a *good configuration* . Figure 4.4 shows three good configurations.

We now present two key theorems that comprise the proof of the four-color theorem.

4.2Theorem *If T is a minimal counterexample, then no good configuration appears in T .*

4.3Theorem *For every internally 6-connected triangulation T , some good configuration appears in T .*

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