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Idempotent Elements of the Semigroups $B_X(D)$ defined by Semilattices of the Class $\Sigma_2(X,8)$, when $Z_7 \cap Z_6 \neq \varnothing$

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ABSTRACT. The paper gives description of idempotent elements of the semigroup $B_X(D)$ which are defined by semilattices of the class $\sum_2(X,8)$, for which intersection the minimal elements is not empty. When X is a finite set, the formulas are derived, by means of which the number of idempotent elements of the semigroup is calculated.

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Key words: *semilattice, semigroup, binary relation, idempotent element.*

Introduction

Let X be an arbitrary nonempty set, D be a $X-$semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such α_f $(f: X\to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a $X-$semilattice of unions D (see ([1], Item 2.1).

By $\varnothing$ we denote an empty binary relation or empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$ and $t\in\breve{D}=\bigcup_{Y\in D} Y$. Then by symbols we denote the following sets:

$$y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y} y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},$$
$$X^*=\{T \mid \varnothing\neq T\subseteq X\},\ D'_t=\{Z'\in D' \mid t\in Z'\},\ Y_T^\alpha=\{x\in X \mid x\alpha=T\},$$
$$D'_T=\{Z'\in D' \mid T\subseteq Z'\},\ \ddot{D}'_T=\{Z'\in D' \mid Z'\subseteq T\}.$$

By symbol $\wedge(D,D_t)$ we mean an exact lower bound of the set D' in the semilattice D.

Definition 1.1. Let $\varepsilon\in B_X(D)$. If $\varepsilon\circ\varepsilon=\varepsilon$, then ε is called an idempotent element of the semigroup $B_X(D)$ and ε is called right unit if $\alpha\circ\varepsilon=\alpha$ for any $\alpha\in B_X(D)$ (see [1], [2], [3]).

Definition 1.2. We say that a complete $X-$semilattice of unions D is an $XI-$semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D,D_t)\in D$ for any $t\in\breve{D}$;

b) $Z=\bigcup_{t\in Z}\wedge(D,D_t)$ for any nonempty element Z of D. (see ([1], definition 1.14.2), ([2] definition 1.14.2), [3], or [4]).

Definition 1.3. Let $\alpha \in B_X(D)$, $T \in V(X^*, \alpha)$ and $Y_T^\alpha = \{y \in X \mid y\alpha = T\}$. A representation of a binary relation α of the form $\alpha = \bigcup_{T \in V(X^*,\alpha)} (Y_T^\alpha \times T)$ is colled quasinormal.

Note that, if $\alpha = \bigcup_{T \in V(X^*,\alpha)} (Y_T^\alpha \times T)$ is a quasinormal representation of a binary relation α, then the following conditions are true:

1) $X = \bigcup_{T \in V(X^*,\alpha)} Y_T^\alpha$;

2) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$, for $T, T' \in V(X^*, \alpha)$ and $T \neq T'$;

Let $\sum_n (X, m)$ denote the class of all complete $X-$semilattice of unions where every element is isomorphic to a fixed semilattice D (see [1]).

Theorem 1.1. Let D be a complete $X-$semilattice of unions. The semigroup $B_X(D)$ possesses right unit iff D is an $XI-$semilattice of unions (see ([1], Theorem 6.1.3), ([2] Theorem6.1.3), or [5]).

Theorem 1.2. Let X be a finite set and $D(\alpha)$ be the set of all those elements T of the semilattice $Q = V(D,\alpha) \setminus \{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. A binary relation α having a quasinormal representation $\alpha = \bigcup_{T \in V(D,\alpha)} (Y_T^\alpha \times T)$ is an idempotent element of this semigroup iff

a) $V(D,\alpha)$ is complete $XI-$semilattise of unions;

b) $\bigcup_{T' \in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq T$ for any $T \in D(\alpha)$;

c) $Y_T^\alpha \cap T \neq \varnothing$ for any nonlimiting element of the set $\ddot{D}(\alpha)_T$ (see ([1], Theorem 6.3.9), ([2], Theorem 6.3.9) or [5]).

Theorem 1.3. Let D, $\Sigma(D)$, $E_X^{(r)}(D')$ and I denote respectively the complete $X-$semilattice of unions, the set of all $XI-$subsemilatices of the semilattice D, the set of all right units of the semigroup $B_X(D')$ and the set of all idempotents of the semigroup $B_X(D)$. Then for the sets $E_X^{(r)}(D')$ and I the following statements are true:

b) if $\varnothing \notin D$, then

1) $E_X^{(r)}(D') \cap E_X^{(r)}(D'') = \varnothing$ for any elements D' and D'' of the set $\Sigma(D)$ that satisfy the condition $D' \neq D''$;

2) $I = \bigcup_{D' \in \Sigma(D)} E_X^{(r)}(D')$;

3) the equality $|I| = \sum_{D' \in \Sigma(D)} \left|E_X^{(r)}(D')\right|$ is fulfilled for the finite set X (see ([1], statement b) Theorem6.2.3), ([2] statement b) Theorem6.2.3), or [5]).

By the symbol $\Sigma_2(X,8)$ we denote the class of all $X-$semilattices of unions whose every element is isomorphic to an $X-$semilattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where

$$\begin{array}{lllll} Z_6 \subset Z_3 \subset Z_1 \subset \breve{D}, & Z_6 \subset Z_4 \subset Z_1 \subset D, & Z_6 \subset Z_4 \subset Z_2 \subset \breve{D}, & & \\ Z_7 \subset Z_4 \subset Z_1 \subset \breve{D}, & Z_7 \subset Z_4 \subset Z_2 \subset \breve{D}, & Z_7 \subset Z_5 \subset Z_2 \subset \breve{D}, & & \\ Z_1 \setminus Z_2 \neq \varnothing, & Z_2 \setminus Z_1 \neq \varnothing, & Z_3 \setminus Z_4 \neq \varnothing, & Z_4 \setminus Z_3 \neq \varnothing, & Z_3 \setminus Z_5 \neq \varnothing, \\ Z_5 \setminus Z_3 \neq \varnothing, & Z_4 \setminus Z_5 \neq \varnothing, & Z_5 \setminus Z_4 \neq \varnothing, & Z_6 \setminus Z_7 \neq \varnothing, & Z_7 \setminus Z_6 \neq \varnothing. \end{array} \tag{1}$$

The semilattice satisfying the conditions (1) is shown in Figure 1. Let $C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ is a family sets, where $P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7$ are pairwise disjoint subsets of the set X and

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$$\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7\\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7\end{pmatrix}$$

is a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

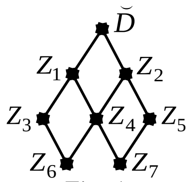

Fig. 1

Diagram of the Semilattice D.

$$\begin{aligned}
\breve{D} &= P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_1 &= P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_2 &= P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_3 &= P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_4 &= P_0\cup P_3\cup P_5\cup P_6\cup P_7\\
Z_5 &= P_0\cup P_1\cup P_3\cup P_4\cup P_6\cup P_7\\
Z_6 &= P_0\cup P_5\cup P_7\\
Z_7 &= P_0\cup P_3\cup P_6
\end{aligned} \qquad (2)$$

here the elements P_1,P_2,P_3,P_5 are basis sources, the element P_0,P_4,P_6,P_7 is sources of completenes of the semilattice D. Therefore $|X|\geq 4$ and $\delta=4$ (see ([1], Item 11.4), ([2], Item 11.4) or [3]).

Now assume that $D\in\Sigma_2(X,8)$. We introduce the following notation:

1) $Q_1=\{T\}$, where $T\in D$ (see diagram 1 in figure 2);

2) $Q_2=\{T,T'\}$, where $T,T'\in D$ and $T\subset T'$ (see diagram 2 in figure 2);

3) $Q_3=\{T,T',T''\}$, where $T,T',T''\in D$ and $T\subset T'\subset T''$ (see diagram 3 in figure 2);

4) $Q_4=\{T,T',T'',\breve{D}\}$, where $T,T',T''\in D$ and $T\subset T'\subset T''\subset\breve{D}$ (see diagram 4 in figure 2);

5) $Q_5=\{T,T',T'',T'\cup T''\}$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$ and $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ (see diagram 5 in figure 2);

6) $Q_6=\{T,Z_4,Z,Z',\breve{D}\}$, where $T\in\{Z_7,Z_6\}$, $Z,Z'\in\{Z_2,Z_1\}$, $Z\neq Z'$, $Z\setminus Z'\neq\varnothing$, $Z'\setminus Z\neq\varnothing$ (see diagram 6 in figure 2);

7) $Q_7=\{T,T',T'',T'\cup T'',\breve{D}\}$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$ and $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ (see diagram 7 in figure 2);

8) $Q_8=\{T,T',Z_4,Z_4\cup T',Z,\breve{D}\}$, where $T\in\{Z_7,Z_6\}$, $T'\in\{Z_5,Z_3\}$, $Z_4\cup T'$, $Z\in\{Z_2,Z_1\}$, $Z_4\cup T'\neq Z$, $T\subset T'$ and $T'\setminus Z_4\neq\varnothing$, $Z_4\setminus T'\neq\varnothing$, $(Z_4\cup T')\setminus Z\neq\varnothing$, $Z\setminus(Z_4\cup T')\neq\varnothing$ (see diagram 8 in figure 2);

9) $Q_9=\{T,T',T\cup T'\}$, where $T,T'\in D$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$ and $T\cap T'=\varnothing$ (see diagram 9 in figure 2);

10) $Q_{10}=\{T,T',T\cup T',T''\}$, where $T,T',T''\in D$, $T\cup T'\subset T''$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$ and $T\cap T'=\varnothing$ (see diagram 10 in figure2);

11) $Q_{11}=\{Z_7,Z_6,Z_4,Z,\breve{D}\}$, where $Z\in\{Z_2,Z_1\}$ and $Z_7\cap Z_6=\varnothing$ (see diagram 11 in figure 2);

12) $Q_{12}=\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\}$, where $Z_7\cap Z_6=\varnothing$ (see diagram 12 in figure 2);

13) $Q_{13}=\{T,T',T\cup T',T'',Z\}$, where $T,T',T'',Z\in D$, $(T\cup T')\subset Z$, $T'\subset T''\subset Z$, $(T\cup T')\setminus T''\neq\varnothing$, $T''\setminus(T\cup T')\neq\varnothing$ and $T\cap T''=\varnothing$ (see diagram 13 in figure 2);

14) $Q_{14}=\{T,T',Z_4,Z,Z',\breve{D}\}$, where $T,T',Z,Z'\in D$, $(T\cup T')\subset Z'$, $T'\subset Z\subset Z'\subset\breve{D}$, $Z_4\setminus Z\neq\varnothing$, $Z\setminus Z_4\neq\varnothing$ and $T\cap Z=\varnothing$ (see diagram 14 in figure 2);

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15) $Q_{15}=\{T',T,Z_4,T'',Z,T''\cup Z_4,\breve{D}\}$, where $T,T'\in\{Z_7,Z_6\}$, $T\neq T'$, $T\subset T''$, $T''\in\{Z_5,Z_3\}$, $Z_4\subset Z$, $Z\cup T''\cup Z_4=\breve{D}$, $(T''\cup Z_4)\setminus Z\neq\varnothing$, $Z\setminus(T''\cup Z_4)\neq\varnothing$, $T''\setminus Z_4\neq\varnothing$, $Z_4\setminus T''\neq\varnothing$ and $T'\cap T''=\varnothing$ (see diagram 15 in figure 2);

16) $Q_{16}=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, where $Z_5\cap Z_3=\varnothing$ (see diagram 16 in figure 2).

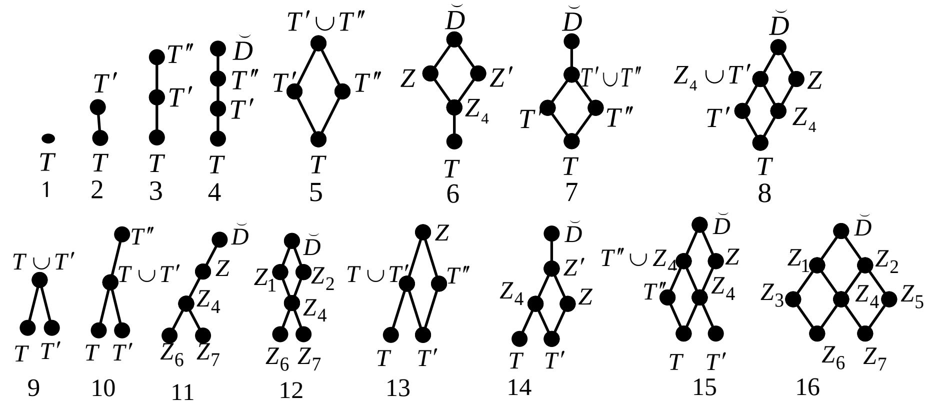

Fig.2

Diagrams of all XI – subsemilattice of the semilattice D.

Denote by the symbol $\Sigma(Q_i)$ $(i=1,2,...,16)$ the set of all XI-subsemilattices of the semilattice D isomorphic to Q_i. Assume that $D'\in\Sigma(Q_i)$ and denote by the symbol $I(D')$ the set of all idempotent elements α of the semigroup $B_X(D')$, for which the semilattices $V(D,\alpha)$ and Q_i are mutually α isomorphic and $V(D,\alpha)=Q_i$.

Definition 1.4. Let the symbol $\Sigma'_{XI}(X,D)$ denote the set of all XI-subsemilattices of the semilattice D.

Let, further, $D,D'\in\Sigma'(X,D)$ and $\vartheta_{XI}\subseteq\Sigma'_{XI}(X,D)\times\Sigma'_{XI}(X,D)$. It is assumed that $D\vartheta_{XI}D'$ if and only if there exists some complete isomorphism φ between the semilattices D and D'. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{XI}(X,D)$.

Let D' be an XI-subsemilattices of the semilattice . By $I(D')$ we denoted the set of all idempotent elements of the semigroup $B_X(D')$ and $\left|I^*(Q_i)\right|=\sum_{D'\in Q_i\vartheta_{XI}}\left|I(D')\right|$, where $i=1,2,...,16$.

Results

Lemma 2.1. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_7\cap Z_6\neq\varnothing$. Then the following sets exhibit all XI – subsemilattices of the considered semilattice D:

1) $\{\breve{D}\},\{Z_1\},\{Z_2\},\{Z_3\},\{Z_4\},\{Z_5\},\{Z_6\},\{Z_7\}$ (see diagram 1 of the figure 2);

2) $\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_4\},\{Z_6,Z_3\},\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},$
$\{Z_5,Z_2\},\{Z_5,\breve{D}\},\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,Z_1\},$ $\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}$
(see diagram 2 of the figure 2);

3) $\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,\breve{D}\},\{Z_7,Z_2,\breve{D}\},\{Z_7,Z_1,\breve{D}\},$
$\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_2,\breve{D}\},\{Z_6,Z_3,Z_1\},\{Z_6,Z_3,\breve{D}\},\{Z_6,Z_1,\breve{D}\},$
$\{Z_5,Z_2,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\}$
(see diagram 3 of the figure 2);

4) $\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\}$ (see diagram 4 of the figure 2);

5) $\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1\},$
$\{Z_6,Z_3,Z_2,\breve{D}\},\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\}$ (see diagram 5 of the figure 2);

6) $\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}$ (see diagram 6 of the figure 2);

7) $\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ (see diagram 7 of the figure 2);

8) $\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (see diagram 8 of the figure 2);

Proof. The statements 1)-4) immediately follows from the Theorems 11.6.1 in [1], the statements 5)-7) immediately follows from the Theorems 11.6.3 in [1] and the statement 8) immediately follows from the Theorems 11.7.2 in [1].

The Lemma is proved.

Lemma 2.2. If $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, then the following equalities are true:

1) $|I(Q_1)|=1$;

2) $|I(Q_2)|=\left(2^{|T'\setminus T|}-1\right)\cdot 2^{|X\setminus T'|}$;

3) $|I(Q_3)|=\left(2^{|T'\setminus T|}-1\right)\cdot\left(3^{|T''\setminus T'|}-2^{|T''\setminus T'|}\right)\cdot 3^{|X\setminus T''|}$;

4) $|I(Q_4)|=\left(2^{|T'\setminus T|}-1\right)\cdot\left(3^{|T''\setminus T'|}-2^{|T''\setminus T'|}\right)\cdot\left(4^{|\breve{D}\setminus T''|}-3^{|\breve{D}\setminus T''|}\right)\cdot 4^{|X\setminus\breve{D}|}$;

5) $|I(Q_5)|=\left(2^{|T'\setminus T''|}-1\right)\cdot\left(2^{|T''\setminus T'|}-1\right)\cdot 4^{|X\setminus(T'\cup T'')|}$;

6) $|I(Q_6)|=\left(2^{|Z_4\setminus T|}-1\right)\cdot 2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(3^{|Z\setminus Z'|}-2^{|Z\setminus Z'|}\right)\cdot\left(3^{|Z'\setminus Z|}-2^{|Z'\setminus Z|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

7) $|I(Q_7)|=\left(2^{|T'\setminus T''|}-1\right)\cdot\left(2^{|T''\setminus T'|}-1\right)\cdot\left(5^{|\breve{D}\setminus(T'\cup T'')|}-4^{|\breve{D}\setminus(T'\cup T'')|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

8) $|I(Q_8)|=\left(2^{|T'\setminus Z|}-1\right)\cdot\left(2^{|Z_4\setminus T'|}-1\right)\cdot\left(3^{|Z\setminus(Z_4\cup T')|}-2^{|Z\setminus(Z_4\cup T')|}\right)\cdot 6^{|X\setminus\breve{D}|}$;

Proof: The statements 1)-4) immediately follows from the Corollary 13.1.5 in [1], the statements 5)-7) immediately follows from the Theorems 13.3.3, the statement 8) immediately follows from the Theorems 13.7.1.

The Lemma is proved.

Theorem 2.1. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$ and $\alpha\in B_X(D)$. The binary relation α is an idempotent relation of the semigroup $B_X(D)$ iff the binary relation α satisfies one of the following conditions:

1) $\alpha=X\times T$, where $T\in D$;

2) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')$, where $T,T'\in D$, $T\subset T'$, $Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\cap T'\neq\varnothing$;

3) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')$, where $T,T',T''\in D$, $T\subset T'\subset T''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

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4) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_0^\alpha\times \breve{D}\right)$, where $T,T',T''\in D$, $T\subset T'\subset T''\subset \breve{D}$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

5) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)$, where $T,T',T''\in D,$ $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing,$ $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

6) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_{Z'}^\alpha\times Z'\right)\cup\left(Y_Z^\alpha\times Z\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_7,Z_6\}$, $Z,Z'\in\{Z_2,Z_1\}$, $Z\neq Z'$, $Z\setminus Z'\neq\varnothing$, $Z'\setminus Z\neq\varnothing$, $Y_T^\alpha, Y_4^\alpha, Y_{Z'}^\alpha, Y_Z^\alpha\notin\{\varnothing\}$ and satisfies the conditions $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z'$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_Z^\alpha\cap Z\neq\varnothing$, $Y_{Z'}^\alpha\cap Z'\neq\varnothing$;

7) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$ $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing,$ $Y_{T''}^\alpha\cap T''\neq\varnothing,$ $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

8) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_{T'\cup Z_4}^\alpha\times\left(T'\cup Z_4\right)\right)\cup\left(Y_Z^\alpha\times Z\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_7,Z_6\}$, $T'\in\{Z_5,Z_3\}$, $T\subset T'$, $Z_4\cup T'$, $Z\in\{Z_2,Z_1\}$, $Z_4\cup T'\neq Z$, $T'\setminus Z_4\neq\varnothing$, $Z_4\setminus T'\neq\varnothing$, $\left(Z_4\cup T'\right)\setminus Z\neq\varnothing$, $Z\setminus\left(Z_4\cup T'\right)\neq\varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_4^\alpha, Y_Z^\alpha\notin\{\varnothing\}$ and satisfies the conditions $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z,$ $Y_{T'}^\alpha\cap T'\neq\varnothing,$ $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_Z^\alpha\cap Z\neq\varnothing$.

Proof: In this case, when $Z_7\cap Z_6\neq\varnothing$, by Lemma 2.4 we know that diagrams 1-8 given in Fig.1 exhibit all diagrams of $XI-$subsemilattices of the semilattices D, a quasinormal representation of idempotent elements of the semigroup $B_X(D)$, which are defined by these $XI-$semilattices, may have one of the forms listed above. The statements 1)-4) immediately follows from the Corollary 13.1.1 in [1], the statements 5)-7) immediately follows from the Corollary 13.3.1 in [1] and the statement 8) immediately follows from the Theorems 13.7.2.

The Theorem is proved.

Lemma 2.3. Let $D=\left\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_1)\right|$ can be calculated by the formula

$$\left|I^*(Q_1)\right|=8.$$

Proof: By the definition of the semilattice D we have

$$Q_1\vartheta_{XI}=\left\{\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\right\}.$$

If the equalities $D_1'=\{Z_7\}$, $D_2'=\{Z_6\}$, $D_3'=\{Z_5\}$, $D_4'=\{Z_4\}$, $D_5'=\{Z_3\}$, $D_6'=\{Z_2\}$, $D_7'=\{Z_1\}$, $D_8'=\{\breve{D}\}$, are fulfilled, then $\left|I^*(Q_1)\right|=\sum_{i=1}^{8}\left|I(D_i')\right|$ (see Definition 1.4). From this equality and statement 1) of Lemma 2.2 we obtain $\left|I^*(Q_1)\right|=1+1+1+1+1+1+1+1=8$.

The Lemma is proved.

Lemma 2.4. Let $D=\left\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_2)\right|$ can be calculated by the formula

$$\begin{aligned}\left|I^*(Q_2)\right| = {} & \left(2^{\left|\breve{D}\setminus Z_1\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|\breve{D}\setminus Z_2\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|\breve{D}\setminus Z_3\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|\breve{D}\setminus Z_4\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|\breve{D}\setminus Z_5\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|\breve{D}\setminus Z_6\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|\breve{D}\setminus Z_7\right|}-1\right)\cdot 2^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_1\setminus Z_3\right|}-1\right)\cdot 2^{\left|X\setminus Z_1\right|}+\\ & +\left(2^{\left|Z_1\setminus Z_4\right|}-1\right)\cdot 2^{\left|X\setminus Z_1\right|}+\left(2^{\left|Z_1\setminus Z_6\right|}-1\right)\cdot 2^{\left|X\setminus Z_1\right|}+\left(2^{\left|Z_1\setminus Z_7\right|}-1\right)\cdot 2^{\left|X\setminus Z_1\right|}+\left(2^{\left|Z_2\setminus Z_4\right|}-1\right)\cdot 2^{\left|X\setminus Z_2\right|}+\\ & +\left(2^{\left|Z_2\setminus Z_5\right|}-1\right)\cdot 2^{\left|X\setminus Z_2\right|}+\left(2^{\left|Z_2\setminus Z_6\right|}-1\right)\cdot 2^{\left|X\setminus Z_2\right|}+\left(2^{\left|Z_2\setminus Z_7\right|}-1\right)\cdot 2^{\left|X\setminus Z_2\right|}+\left(2^{\left|Z_3\setminus Z_6\right|}-1\right)\cdot 2^{\left|X\setminus Z_3\right|}+\\ & +\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot 2^{\left|X\setminus Z_4\right|}+\left(2^{\left|Z_4\setminus Z_7\right|}-1\right)\cdot 2^{\left|X\setminus Z_4\right|}+\left(2^{\left|Z_5\setminus Z_7\right|}-1\right)\cdot 2^{\left|X\setminus Z_5\right|}\end{aligned}$$

Proof: By the definition of the semilattice D we have

$$\begin{aligned}Q_2\theta_{XI} = {} & \left\{Z_1,\breve{D}\right\},\left\{Z_2,\breve{D}\right\},\left\{Z_3,\breve{D}\right\},\left\{Z_4,\breve{D}\right\},\left\{Z_5,\breve{D}\right\},\left\{Z_6,\breve{D}\right\},\left\{Z_7,\breve{D}\right\},\left\{Z_3,Z_1\right\},\left\{Z_4,Z_1\right\},\left\{Z_6,Z_1\right\},\\ & \left\{Z_7,Z_1\right\},\left\{Z_4,Z_2\right\},\left\{Z_5,Z_2\right\},\left\{Z_6,Z_2\right\},\left\{Z_7,Z_2\right\},\left\{Z_6,Z_3\right\},\left\{Z_6,Z_4\right\},\left\{Z_7,Z_4\right\},\left\{Z_7,Z_5\right\}\}.\end{aligned}$$

If the equalities

$$\begin{aligned}& D_1'=\left\{Z_1,\breve{D}\right\}, D_2'=\left\{Z_2,\breve{D}\right\}, D_3'=\left\{Z_3,\breve{D}\right\}, D_4'=\left\{Z_4,\breve{D}\right\}, D_5'=\left\{Z_5,\breve{D}\right\},\\ & D_6'=\left\{Z_6,\breve{D}\right\}, D_7'=\left\{Z_7,\breve{D}\right\}, D_8'=\left\{Z_3,Z_1\right\}, D_9'=\left\{Z_4,Z_1\right\}, D_{10}'=,\left\{Z_6,Z_1\right\},\\ & D_{11}'=,\left\{Z_7,Z_1\right\}, D_{12}'=\left\{Z_4,Z_2\right\}, D_{13}'=\left\{Z_5,Z_2\right\}, D_{14}'=\left\{Z_6,Z_2\right\}, D_{15}'=\left\{Z_7,Z_2\right\},\\ & D_{16}'=\left\{Z_6,Z_3\right\}, D_{17}'=\left\{Z_6,Z_4\right\}, D_{18}'=\left\{Z_7,Z_4\right\}, D_{19}'=\left\{Z_7,Z_5\right\}.\end{aligned}$$

are fulfilled, then

$$\left|I^*(Q_2)\right| = \sum_{i=1}^{19}\left|I\left(D_i'\right)\right|$$

(see Definition 1.4). From this equality and the statement 2) of Lemma 2.2 we obtain the validity of Lemma 2.4.

The Lemma is proved.

Lemma 2.5. Let $D=\left\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_3)\right|$ can be calculated by the formula

$$\begin{aligned}\left|I^*(Q_3)\right| = {} & \left(2^{\left|Z_1\setminus Z_3\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_1\setminus Z_4\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|Z_2\setminus Z_4\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_2\setminus Z_5\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|Z_1\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_2\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|Z_3\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_4\right|}-2^{\left|\breve{D}\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|Z_1\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_2\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|Z_4\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_4\right|}-2^{\left|\breve{D}\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\left(2^{\left|Z_5\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_5\right|}-2^{\left|\breve{D}\setminus Z_5\right|}\right)\cdot 3^{\left|X\setminus \breve{D}\right|}+\\ & +\left(2^{\left|Z_3\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|Z_1\setminus Z_3\right|}-2^{\left|Z_1\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus Z_1\right|}+\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|Z_1\setminus Z_4\right|}-2^{\left|Z_1\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus Z_1\right|}+\\ & +\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|Z_2\setminus Z_4\right|}-2^{\left|Z_2\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus Z_2\right|}+\left(2^{\left|Z_4\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|Z_1\setminus Z_4\right|}-2^{\left|Z_1\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus Z_1\right|}+\\ & +\left(2^{\left|Z_4\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|Z_2\setminus Z_4\right|}-2^{\left|Z_2\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus Z_2\right|}+\left(2^{\left|Z_5\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|Z_2\setminus Z_5\right|}-2^{\left|Z_2\setminus Z_5\right|}\right)\cdot 3^{\left|X\setminus Z_2\right|}\end{aligned}$$

Proof: By the definition of the semilattice D we have

$$\begin{aligned}Q_3\theta_{XI} = {} & \left\{\left\{Z_3,Z_1,\breve{D}\right\},\left\{Z_4,Z_1,\breve{D}\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_5,Z_2,\breve{D}\right\},\left\{Z_6,Z_1,\breve{D}\right\},\left\{Z_6,Z_2,\breve{D}\right\},\right.\\ & \left\{Z_6,Z_3,\breve{D}\right\},\left\{Z_6,Z_4,\breve{D}\right\},\left\{Z_7,Z_1,\breve{D}\right\},\left\{Z_7,Z_2,\breve{D}\right\},\left\{Z_7,Z_4,\breve{D}\right\},\left\{Z_7,Z_5,\breve{D}\right\},\\ & \left.\left\{Z_6,Z_3,Z_1\right\},\left\{Z_6,Z_4,Z_1\right\},\left\{Z_6,Z_4,Z_2\right\},\left\{Z_7,Z_4,Z_1\right\},\left\{Z_7,Z_4,Z_2\right\},\left\{Z_7,Z_5,Z_2\right\}\right\}.\end{aligned}$$

If the equalities

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$$D_1'=\{Z_3,Z_1,\breve{D}\},D_2'=\{Z_4,Z_1,\breve{D}\},D_3'=\{Z_4,Z_2,\breve{D}\},D_4'=\{Z_5,Z_2,\breve{D}\},D_5'=\{Z_6,Z_1,\breve{D}\},$$
$$D_6'=\{Z_6,Z_2,\breve{D}\},D_7'=\{Z_6,Z_3,\breve{D}\},D_8'=\{Z_6,Z_4,\breve{D}\},D_9'=\{Z_7,Z_1,\breve{D}\},D_{10}'=\{Z_7,Z_2,\breve{D}\},$$
$$D_{11}'=\{Z_7,Z_4,\breve{D}\},D_{12}'=\{Z_7,Z_5,\breve{D}\},D_{13}'=\{Z_6,Z_3,Z_1\},D_{14}'=\{Z_6,Z_4,Z_1\},D_{15}'=\{Z_6,Z_4,Z_2\},$$
$$D_{16}'=\{Z_7,Z_4,Z_1\},D_{17}'=\{Z_7,Z_4,Z_2\},D_{18}'=\{Z_7,Z_5,Z_2\}.$$

are fulfilled, then

$$\left|I^*(Q_3)\right|=\sum_{i=1}^{18}\left|I(D_i')\right|$$

(see Definition 1.4). From this equality and the statement 3) of Lemma 2.2 we obtain the validity of Lemma 2.5.

The Lemma is proved.

Lemma 2.6. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_4)\right|$ can be calculated by the formula

$$\begin{aligned}\left|I^*(Q_4)\right|&=\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot4^{|X\setminus\breve{D}|}+\\&+\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot4^{|X\setminus\breve{D}|}+\\&+\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot4^{|X\setminus\breve{D}|}+\\&+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot4^{|X\setminus\breve{D}|}+\\&+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot4^{|X\setminus\breve{D}|}+\\&+\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot4^{|X\setminus\breve{D}|}\end{aligned}$$

Proof: By the definition of the semilattice D we have

$$\begin{aligned}Q_4\theta_{XI}=\{&\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\}\\&\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\}\}\end{aligned}$$

If the equalities

$$D_1'=\{Z_7,Z_5,Z_2,\breve{D}\},D_2'=\{Z_7,Z_4,Z_2,\breve{D}\},D_3'=\{Z_7,Z_4,Z_1,\breve{D}\},$$
$$D_4'=\{Z_6,Z_4,Z_2,\breve{D}\},D_5'=\{Z_6,Z_4,Z_1,\breve{D}\},D_6'=\{Z_6,Z_3,Z_1,\breve{D}\},$$

are fulfilled, then

$$\left|I^*(Q_4)\right|=\sum_{i=1}^{6}\left|I(D_i')\right|$$

(see Definition 1.4). From this equality and the statement 4) of Lemma 2.2 we obtain the validity of Lemma 2.6.

The Lemma is proved.

Lemma 2.7. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_5)\right|$ can be calculated by the formula

$$\begin{aligned}\left|I^*(Q_5)\right|&=3\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot4^{|X\setminus\breve{D}|}+\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot4^{|X\setminus Z_2|}+\\&+\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_5|}-1\right)\cdot4^{|X\setminus\breve{D}|}+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot4^{|X\setminus Z_1|}+\\&+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot4^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By the definition of the semilattice D we have

$$\begin{aligned}Q_5\theta_{XI}=\{&\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2\}\\&\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1\},\{Z_6,Z_3,Z_2,\breve{D}\}\}\end{aligned}$$

If the equalities

$$D_1'=\{Z_7,Z_2,Z_1,\breve{D}\},D_2'=\{Z_6,Z_2,Z_1,\breve{D}\},D_3'=\{Z_4,Z_2,Z_1,\breve{D}\},D_4'=\{Z_7,Z_5,Z_4,Z_2\},$$
$$D_5'=\{Z_7,Z_5,Z_1,\breve{D}\},D_6'=\{Z_6,Z_4,Z_3,Z_1\},D_7'=\{Z_6,Z_3,Z_2,\breve{D}\}$$

are fulfilled, then

$$\left|I^*(Q_5)\right|=\sum_{i=1}^{7}\left|I(D_i')\right|$$

(see Definition 1.4). From this equality and the statement 5) of Lemma 2.2 we obtain the validity of Lemma 2.7.

The Lemma is proved.

Lemma 2.8. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_6)\right|$ can be calculated by the formula

$$\left|I^*(Q_6)\right|=\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}+$$
$$+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$

Proof: By the definition of the semilattice D we have $Q_6\theta_{XI}=\{\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}\}$.

If the equalities $D_1'=\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},D_2'=\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}$ are fulfilled, then

$$\left|I^*(Q_6)\right|=\left|I(D_1')\right|+\left|I(D_2')\right|$$

(see Definition 1.4). From this equality and the statement 6) of Lemma 2.2 we obtain the validity of Lemma 2.8.

The Lemma is proved.

Lemma 2.9. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_7)\right|$ can be calculated by the formula

$$\left|I^*(Q_7)\right|=\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}+$$
$$+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$

Proof: By the definition of the semilattice D we have $Q_7\theta_{XI}=\{\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}\}$.

If the equalities $D_1'=\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},D_2'=\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ are fulfilled, then

$$\left|I^*(Q_7)\right|=\left|I(D_1')\right|+\left|I(D_2')\right|$$

(see Definition 1.4). From this equality and the statement 7) of Lemma 2.2 we obtain the validity of Lemma 2.9.

The Lemma is proved.

Lemma 2.10. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$, $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_8)\right|$ can be calculated by the formula

$$\left|I^*(Q_8)\right|=\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}+$$
$$+\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}$$

Proof: By the definition of the semilattice D we have $Q_8\theta_{XI}=\{\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\}$.

If the equalities $D_1'=\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\},D_2'=\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ are fulfilled, then

$$\left|I^*(Q_8)\right|=\left|I(D_1')\right|+\left|I(D_2')\right|$$

(see Definition 1.4). From this equality and the statement 8) of Lemma 2.2 we obtain the validity of Lemma 2.10.

The Lemma is proved.

Let us assume that

$$k_1 = \sum_{i=1}^{8} \left| I^*(Q_i) \right|$$

Theorem 2.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set and I_D is a set of all idempotent elements of the semigroup $B_X(D)$, then $|I_D| = k_1$.

Proof: This Theorem immediately follows from the Theorem 2.1.

The Theorem is proved.

Example 2.1. Let $X = \{1,2,3,4,5\}$, $P_0 = \{1\}$, $P_1 = \{2\}$, $P_2 = \{3\}$, $P_3 = \{4\}$, $P_5 = \{5\}$, $P_4 = P_6 = P_7 = \varnothing$.

Then $\breve{D} = \{1,2,3,4,5\}$, $Z_1 = \{1,3,4,5\}$, $Z_2 = \{1,2,4,5\}$, $Z_3 = \{1,3,5\}$, $Z_4 = \{1,4,5\}$, $Z_5 = \{1,2,4\}$, $Z_6 = \{1,5\}$, $Z_7 = \{1,4\}$ and

$$D = \{\{1,4\},\{1,5\},\{1,2,4\},\{1,4,5\},\{1,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{1,2,3,4,5\}\}.$$

Therefore we have that following equality and inequality is valid:

$$Z_7 \cap Z_6 = \{1,4\} \cap \{1,5\} \neq \varnothing,$$

where $|I^*(Q_1)| = 8$, $|I^*(Q_2)| = 73$, $|I^*(Q_3)| = 54$, $|I^*(Q_4)| = 6$, $|I^*(Q_5)| = 17$, $|I^*(Q_6)| = 2$, $|I^*(Q_7)| = 2$, $|I^*(Q_8)| = 2$, $|I_D| = 164$.

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