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Inner and Outer Regularity of a Measure

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Abstract: If μ be a measure on the σ-algebra of Borel sets of a Hausdorff topological space *X*. Then measure μ is called inner regular or tight if, for any Borel set *B*, μ (*B*) is the supremum of μ (*K*) over all compact subsets *K* of *B*. The measure μ is called outer regular if, for any Borel set *B*, μ (*B*) is the infimum of *m* μ*U*) over all open sets *U* containing *B*. Here we prove that *the measure* μ is called Regular if it is inner regular and as well as outer regular and proves some of the properties of inner regularity and outer regularity.

Definition: Let X be any set, be $\mathcal{S}$ any σ-ring on X, $\mathcal{C}$ and $\mathcal{U}$ be any subclasses of $\mathcal{S}$.

(1) μ be any measure on $\mathcal{S}$ i.e. (X, $\mathcal{S},\mu$) is a measure space then we say (X, $\mathcal{S},\mu,\mathcal{C},\mathcal{U}$) satisfies axiom I.

(2) *If* $\mathcal{C}$ is closed for finite unions, countable intersections, $\phi \in \mathcal{C}$ and $\mu(c) < \infty$ ∀ c ∈ $\mathcal{C}$, then we say that axiom II is satisfied. i.e. we say (X, $\mathcal{S},\mu,\mathcal{C},\mathcal{U}$) satisfies axiom II.

(3) *If* $\mathcal{U}$ is clsed for countable unions, finite intersections and for every E ∈ $\mathcal{S}$ there exist U ∈ $\mathcal{U}$ s.t. E ⊂ U,then we say that (X, $\mathcal{S},\mu,\mathcal{C},\mathcal{U}$) satisfies axiom III.

Definition: Suppose that (X, $\mathcal{S},\mu,\mathcal{C},\mathcal{U}$) satisfy axiom I, II and III. Let E ∈ $\mathcal{S}$

(1) If $\mu(E)$ = inf{ $\mu(U)$/ E ⊂ U ∈ $\mathcal{U}$}, then E is said to be Outer regular.

(2) If $\mu(E)$ = sup{ $\mu(C)$/ E ⊃ C ∈ $\mathcal{C}$ }, then E is said to be Inner regular.

(3) The set E is said to be Regular if it is Outer regular as well as Inner regular.

(4) The measure μ is called Regular if every measurable set E in $\mathcal{S}$ is Regular.

Preposition: Suppose that (X, $\mathcal{S},\mu,\mathcal{C},\mathcal{U}$) satisfies axiom I, II and III. Let E ∈ $\mathcal{S}$ then

(1) E is Outer regular iff for every $\varepsilon > 0$ there exist U ∈ $\mathcal{U}$ s.t. E ⊂ U and $\mu(U) \leq \mu(E) + \varepsilon$.

(2) If for every $\varepsilon > 0$ there exist C ∈ $\mathcal{C}$ s.t. C ⊂ E and $\mu(E) \leq \mu(C) + \varepsilon$, then E is called Inner regular.

(3) If E is Inner regular and $\mu(E) < \infty$ then for each $\varepsilon > 0$ there exist C ∈ $\mathcal{C}$, s.t. C ⊂ E and $\mu(E) \leq \mu(C) + \varepsilon$.

Proof: (1) Suppose E is Outer regular and $\varepsilon > 0$.

Let $\mu(E) = \infty$, By axiom III there exist U ∈ $\mathcal{U}$ s.t. E ⊂ U ⇒ $\mu(U) \geq \mu(E) = \infty$
⇒ $\mu(U) = \infty$ ⇒ $\mu(U) \leq \mu(E) + \varepsilon$.

Now suppose that $\mu(E) < \infty$, then we have $\mu(E) \leq \mu(E) + \varepsilon$ and
$\mu(E)$ = inf{ $\mu(U)$/ E ⊂ U ∈ $\mathcal{U}$} then by definition of infimum there exist U ∈ $\mathcal{U}$ s.t. E ⊂ U and $\mu(U) \leq \mu(E) + \varepsilon$, shows that the condition is necessary.

Conversely: Assume that the condition is satisfied.

To show that E is Outer regular, Let n be any natural number, By the condition taking $\varepsilon = \frac{1}{n}$ we have $U_n \in \mathcal{U}$ *s.t.* E ⊂ U_n and $\mu(U_n) \leq \mu(E) + \frac{1}{n}$.

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Let $V_n = \bigcap_{i=1}^{n} U_i$ then $V_n \in \mathcal{U}$, (V_n) is a decreasing sequence and

$\mu(E)+\frac{1}{n} \geq \mu(U_n) \geq \mu(V_n) \quad \forall$ n

Therefore $\lim_{n\to\infty}\{\mu(E)+\frac{1}{n}\} \geq \lim_{n\to\infty} \mu(U_n)$

$\Rightarrow \mu(E) \geq \inf\{\mu(U_n)\} \geq \inf\{\mu(V_n)/ E \subset V, \quad V \in \mathcal{U}\}$ ……….(*)

On the other hand $\mu(E) \leq \mu(V)$ for all V s.t. $E \subset V, \quad V \in \mathcal{U}$.

$\Rightarrow \mu(E) \leq \inf\{\mu(V)/E \subset V, \quad V \in \mathcal{U}\}$ …………(**)

From (*) and (**) we have $\mu(E) = \inf\{\mu(V)/E \subset V, \quad V \in \mathcal{U}\}$, shows that E is Outer regular.

(2) And (3) can be proved similarly by using the definition of Supremum.

Preposition: (1) If $\mu(E) = \infty$ then E is Outer regular.

(2) Every member of $\mathcal{U}$ is Outer regular.

(3) If $V = \bigcap_{n=1}^{\infty} U_n$, $U_n \in \mathcal{U}$, $\mu(U) < \infty$ then V is Outer regular.

Proof: (1) Let $\mu(E) = \infty$ and $U \in \mathcal{U}$ s.t. $E \subset U$, then $\mu(U) = \infty$

$\Rightarrow \inf\{\mu(U)/E \subset U, \ U \in \mathcal{U}\} = \infty \Rightarrow \mu(E) = \inf\{\mu(U)/E \subset U, \quad U \in \mathcal{U}\}$. Then E is Outer regular.

(2) Let $W \in \mathcal{U}$ then $\mu(W) \geq \inf\{\mu(U)/W \subset U, \quad U \in \mathcal{U}\}$

Let $U \in \mathcal{U}$ s.t. $W \subset U$ then $(W) \leq \mu(U) \Rightarrow \mu(W) \leq \inf\{\mu(U)/W \subset U, \quad U \in \mathcal{U}\}$

$\Rightarrow \mu(W) = \inf\{\mu(U)/W \subset U, \quad U \in \mathcal{U}\}$, shows that W is Outer regular.

(3) Let (U_n) be any sequence of members of $\mathcal{U}$ s.t. $\mu(U_1) < \infty$

Let $V = \bigcap_{n=1}^{\infty} U_n$, Defne $V_n = \bigcap_{i=1}^{n} U_i$ then $V_n \in \mathcal{U}$ and $(V_n) \downarrow \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n = V$

$\Rightarrow \mu(V) = \lim_{n\to\infty} \mu(V_n) = \inf\{\mu(V_n)\} \geq \inf\{\mu(U)/ \ V \subset U \in \mathcal{U}\}$

$\Rightarrow \mu(V) \geq \inf\{\mu(U)/ \ V \subset U \in \mathcal{U}\}$ … … … . (*)

Let U be any member of $\mathcal{U}$ s.t. $V \subset U$ then $\mu(V) \leq \mu(U)$

$\Rightarrow \mu(V) \leq \inf\{\mu(U)/ \ V \subset U \in \mathcal{U}\}$ …………..(**)

From (*) and (**) we get, $\mu(V) = \inf\{\mu(U)/ \ V \subset U \in \mathcal{U}\}$

Shows that V is Outer regular.

Preposition: (1) If $\mu(E) = 0$ then E is Inner regular.

(2) Every member of $\mathcal{C}$ is Inner regular.

(3) Countable unions of members of $\mathcal{C}$ is Inner regular.

Proof: (1) Let $\mu(E) = 0$ and $C \in \mathcal{C}$ and $C \subset E$ then $\mu(C) \leq \mu(E) \Rightarrow \mu(C) = 0$

$\Rightarrow \text{Sup}\{\mu(C)/ \ C \subset E, C \in \mathcal{C}\} = 0 \Rightarrow \mu(E) = \text{Sup}\{\mu(C)/ \ C \subset E, C \in \mathcal{C}\}$.

Hence E is Inner regular.

(2) Let $D \in \mathcal{C}$ then $D \subseteq D$, and $D \in \mathcal{C}$

Therefore $\text{Sup}\{\mu(C)/ \ C \subset D, C \in \mathcal{C}\} \geq \mu(D)$

Also $C \subset D \Rightarrow \mu(C) \leq \mu(D) \Rightarrow \text{Sup}\{\mu(C)/ \ C \subset D, C \in \mathcal{C}\} \leq \mu(D)$

Therefore $\mu(D) = \text{Sup}\{\mu(C)/ \ C \subset D, C \in \mathcal{C}\} \Rightarrow$ D is Inner regular.

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(3) Let (D_n) be a sequence of members of $\mathcal{C}$ and D = $\bigcup_{n=1}^{\infty} D_n$, Define $C_n = \bigcup_{j=1}^{n} D_j$ then $C_n \in \mathcal{C}$ for all n and (C_n) is monotone increasing sequence with $\bigcup_{n=1}^{\infty}$ C= $\bigcup_{n=1}^{\infty} D_n$= D

$\Rightarrow (C_n) \uparrow D \Rightarrow \mu(C_n) \to \mu(D)$

$\Rightarrow \mu(D) = \lim_{n\to\infty} \mu(C_n) = \text{Sup}\{\mu(C_n)\} \leq \text{Sup}\{\mu(C)/ C \subset D, C \in \mathcal{C}\}$

$\Rightarrow \mu(D) \leq \text{Sup}\{\mu(C)/ C \subset D, C \in \mathcal{C}\}$ On the other hand if $C \subset D$, $C \in \mathcal{C}$ then $\mu(C) \leq \mu(D)$

$\Rightarrow \text{Sup}\{\mu(C)/ C \subset D, C \in \mathcal{C}\} \leq \mu(D) \Rightarrow \mu(D) = \text{Sup}\{\mu(C)/ C \subset D, C \in \mathcal{C}\}$

$\Rightarrow$ D is Inner regular.

Hence the proof.

Theorem: Countable union of outer regular sets is outer regular.

Proof: Let (E_n) be any sequence of outer regular sets and E = $\bigcup_{n=1}^{\infty} E_n$

If $\mu(E) = \infty$, then E is outer regular as proved earlier.

Now suppose that $\mu(E) < \infty$, Let $\varepsilon > 0$, since E_n are outer regular we can find a set $U_n \in \mathcal{U}$ s.t. $E_n \subset U_n$, and $\mu(U_n) \leq \mu(E_n) + \frac{\varepsilon}{2^n}$

Let U = $\bigcup_{n=1}^{\infty} U_n$, then $U \in \mathcal{U}$ and $E \subset U$ then

$$\mu(U - E) = \mu\left(\left(\bigcup_{n=1}^{\infty} U_n\right) - \left(\bigcup_{n=1}^{\infty} E_n\right)\right)$$

$$\leq \mu\left(\bigcup_{n=1}^{\infty} (U_n - E_n)\right) \leq \sum_{n=1}^{\infty} \mu(U_n - E_n) = \sum_{n=1}^{\infty} [\mu(U_n) - \mu(E_n)] \text{ [Because } \mu(E_n) < \infty]$$

$$\leq \sum_{n=1}^{\infty} \left[\frac{\varepsilon}{2^n}\right] = \varepsilon$$

$\Rightarrow \mu(U - E) \leq \varepsilon \Rightarrow \mu(U) \leq \mu(E) + \varepsilon \Rightarrow$ E is outer regular.

Theorem: Finite union of outer regular sets is outer regular.

Proof: Let $E_1, E_2, \ldots\ldots\ldots, E_k$ be k outer regular sets and let E= $\bigcup_{i=1}^{k} E_i$,

Define $E_n = E_k$ for n > k. Then (E_n) is a sequence of outer regular sets and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{k} E_n$= E.

Hence E is outer regular set by the proceeding theorem.

Theorem: Finite intersection of outer regular sets of finite measure is outer regular.

Proof: Suppose E and F are outer regular sets and $\mu(E) < \infty$ *and* $\mu(F) < \infty$.

Let $\varepsilon > 0$ *be* given, by the outer regularity of E, we can find a set $U \in \mathcal{U}$ s.t. $E \subset U$ and $\mu(U) \leq \mu(E) + \frac{\varepsilon}{2}$

Similarly we can find a set $V \in \mathcal{U}$ s.t. $F \subset V$ and $\mu(V) \leq \mu(F) + \frac{\varepsilon}{2}$

Then $U \cap V \in \mathcal{U}$, $E \cap F \subset U \cap V$ and $\mu[(U \cap V) - (E \cap F)] \leq \mu[(U - E) \cup (V - F)]$

$\leq \mu(U - E) + \mu(V - F) = [\mu(U) - \mu(E)] + [\mu(V) - \mu(F)] = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \mu[(U \cap V) - (E \cap F)] \leq \varepsilon \Rightarrow E \cap F$ is outer regular.

Theorem: The countable intersection of outer regular sets of finite measure is outer regular.

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Proof: Let (E_n) be any sequence of outer regular sets of finite measure and $E = \bigcap_{n=1}^{\infty} E_n$.

To show that E is outer regular.

Let $\varepsilon > 0$, Define $F_n = \bigcap_{j=1}^{n} E_j$, then (F_n) is a decreasing sequence of outer regular sets and $\lim_{n\to\infty}(F_n) = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} E_n = E$. Thus $(F_n) \downarrow E$ and $\mu(F_i) < \infty$ for all i.

By continuity of measure for decreasing sequences we obtain that

$\mu(E) = \lim_{n\to\infty} \mu(F_n)$ i.e. $\mu(F_n) \to \mu(E)$

$\Rightarrow$ There exist k s.t. $\mu(F_k) \le \mu(E) + \frac{\varepsilon}{2}$, Since F_k is outer regular and $\mu(F_k) < \infty$, we can find $U \in \mathcal{U}$ s.t. $F_k \subset U$ and $\mu(U) \le \mu(F_k) + \frac{\varepsilon}{2}$,

Thus $E \subset U$ and $\mu(U - E) = \mu[(U - F_k) \cup (F_k - E)]$

$\le \mu(U - F_k) + \mu(F_k - E) \le \mu(U) - \mu(F_k) + \mu(F_k) - \mu(E) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \mu(U - E) < \varepsilon$, Shows that E is outer regular.

Theorem: Finite union of inner regular sets is inner regular.

Proof: Let E and F be two inner regular sets. To show that E∪F is also inner regular.

(1) Let $\mu(E) = \infty$,then $\text{Sup}\{ \mu(C) / E \supset C \in \mathcal{C}\} = \infty$

$\Rightarrow \text{Sup}\{ \mu(C)/C \subset E \cup F, C \in \mathcal{C}\} = \infty$

The fact $\mu(E) = \infty$ gives that $\mu(E \cup F) = \infty \Rightarrow \mu(E \cup F) = \text{Sup}\{ \mu(C)/C \subset E \cup F, C \in \mathcal{C}\}$

$\Rightarrow E \cup F$ is inner regular.

(2) Let $\mu(F) = \infty$, then the argument is same as above.

(3) Finally suppose that $\mu(E) < \infty$, $\mu(F) < \infty$, Consider any $\varepsilon > 0$, as E is inner regular and $\mu(E) < \infty$,therefore there exist $C \in \mathcal{C}$ s.t. $C \subset E$ and $\mu(E) < \mu(C) + \frac{\varepsilon}{2}$

By the same argument there exist D, $D \subset F$ and $\mu(F) < \mu(D) + \frac{\varepsilon}{2}$

Now $C \cup D \in \mathcal{C}$, $C \cup D \subset E \cup F$ and $\mu[(E \cup F) - (C \cup D)] \le \mu[(E - C)] + \mu[(F - D)]$

$\le \mu[(E - C)] + \mu[(F - D)] = \mu(E) - \mu(C) + \mu(F) - \mu(D) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \mu[(E \cup F) - (C \cup D)] \le \varepsilon \Rightarrow \mu(E \cup F) \le \mu(C \cup D) + \varepsilon \Rightarrow E \cup F$ is inner regular.

Theorem: The countable union of Inner regular sets is Inner regular.

Proof: Let (E_n) be any sequence of inner regular sets and $E = \bigcup_{n=1}^{\infty} E_n$,

Let $F_n = \bigcup_{i=1}^{n} E_i$, then in view of the above theorem F_n is inner regular for all n. Also F_n is monotonic increasing sequence and $\lim_{n\to\infty}(F_n) = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n = E$

i.e.$(F_n) \uparrow E \Rightarrow \mu(F_n) \to \mu(E)$ ……………….(*)

Case (1) Suppose $\mu(E) = \infty$. Let n be any natural number.

Since $\mu(F_n) \to \infty$, we can find k s.t. $\mu(F_k) > n$,

As F_k is inner regular, we have $\mu(F_k) = \text{Sup}\{ \mu(C)/ C \subset F_k, C \in \mathcal{C}\}$ and $C \subset F_k \subset E$

$\Rightarrow C \in \mathcal{C}$, $C \subset E$ and $\mu(C) > n \Rightarrow \text{Sup}\{ \mu(C)/ E \supset C \in \mathcal{C}\} = \infty$

$\Rightarrow \mu(E) = \text{Sup}\{ \mu(C)/ E \supset C \in \mathcal{C}\} \Rightarrow$ E is inner regular.

Case(2) *Let* $\mu(E) < \infty$, Then *take* $\varepsilon > 0$, as $\mu(E) < \infty$ and $\mu(F_k) \to \mu(E)$, we can find k s.t.

$\mu(F_k) \leq \mu(E)+\frac{\varepsilon}{2}$, for inner regularity of F_k, we can find D∈ $\mathcal{U}$ s.t. D ⊂ F_k

And $\mu(F_k) < \mu(D_k) + \frac{\varepsilon}{2}$,

$Then\ \mu(E - D) = \mu[(E - F_k) \cup (F_k - D)] = \mu(E) - \mu(F_k) + \mu(F_k) - \mu(D) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \mu(E - D) < \varepsilon \Rightarrow \mu(E) - \mu(D) < \varepsilon \Rightarrow \mu(E) < \mu(D) + \varepsilon$, Proves that E is inner regular.

Theorem: Countable intersection of inner regular sets of finite measure is inner regular.

Proof: Let (E_n) be any sequence of inner regular sets s.t. $\mu(E_n) < \infty, \forall$ n.

Let E = $\bigcap_{n=1}^{\infty} E_n$, Let $\varepsilon > 0$, Since E_n is inner regular and $\mu(E_n) < \infty$, we can find a set $C_n \in \mathcal{C}$ s.t. $C_n \subset E_n$ and $\mu(E_n) < \mu(C_n) + \frac{\varepsilon}{2^n}$, Define C = $\bigcap_{n=1}^{\infty} C_n$, Then C ∈ $\mathcal{C}$, C ⊂ E

and $\mu(E - C) = \mu[\left(\bigcap_{n=1}^{\infty} E_n\right) - \left(\bigcap_{n=1}^{\infty} C_n\right)] \leq \mu\left(\bigcup_{n=1}^{\infty} (E_n - C_n)\right) \leq \sum_{n=1}^{\infty} \mu(E_n - C_n)$

$\leq \sum_{1}^{\infty} \mu(E_n) - \sum_{1}^{\infty} \mu(C_n) \leq \sum_{1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

$\Rightarrow \mu(E) - \mu(C) < \varepsilon \Rightarrow \mu(E) < \mu(C) + \varepsilon \Rightarrow$ E is inner regular.

Theorem: Finite intersection of inner regular sets of finite measure is inner regular.
Proof: Let $E_1, E_2, \ldots \ldots, E_k$ be finitely many inner regular sets with $\mu(E_i) < \infty$ for $1 \leq i \leq k$
Define $E_n = E_k$ for n > k.
Then (E_n) is a sequence of inner regular sets with $\mu(E_n) < \infty$ for all n.
By the proceeding theorem $\bigcap_{n=1}^{\infty} E_n$ is inner regular.
But $\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{k} E_n \Rightarrow \bigcap_{n=1}^{k} E_n$ is inner regular.
Note: From the above said theorems we can say that
(1) Countable union of regular sets is regular.
(2) Finite union of regular sets is regular.
(3) Countable intersection of regular sets of finite measure is regular.
(4) Finite intersection of regular sets of finite measure is regular.

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