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ON G-FUZZY IDEALS

A.R.ALIZADEH M.*

Abstract. One branch with many applications in fuzzy algebra is the application of fuzzy logic in ideals, This article is based on data obtained from papers presented by Alizadeh [1], Swamy[19], Kumbhojkar[15], Zahedi[21], Mukherjee[18] and Zhang Yue[22]. Furthermore we have some new concepts about the G-ideals which are used by the reference of Kaplansky [12]. Finally by the following properties and theorems related to the G-ideals we have presented some definitions and theorems about G-fuzzy ideals.

1. Introduction to Fuzzy ideals

On this section, we introduce some concepts on fuzzy ideals in a commutative ring R and we are point the theorems and corollaries related which are used in the next section. So The R are be using as the commutative and unitary ring and the $F(R)$ is the set of all fuzzy subset and the $I(R)$ is also the set of all fuzzy ideals on R.

Definition 1.1. Let $A \in F(R)$ be any fuzzy subset of R , we call the A is a fuzzy subset of R, if for each $a, b \in R$ we have:
i) $A(a - b) \geq inf\{A(a), A(b)\} = (A(a) \wedge A(b))$.
ii) $A(ab) \geq inf\{A(a), A(b)\} = (A(a) \wedge A(b))$.

Definition 1.2. The $A \in F(R)$ is a fuzzy subdomain of R if:
i) A is a fuzzy subring of R.
ii) $\forall x, y \in R$, if $A(x, y) = 0$, then: $A(x) = 0$ or $A(y) = 0$.
It is obvious that $A(x, y) \geq \{A(x), A(y)\}$ (Since A is a fuzzy subring of R).

Definition 1.3. The set of all fuzzy subdomain of D is called by $F(D)$, and for each $A, B \in F(D)$:
i) $(A \cap B)(x) = inf\{A(x), B(x)\} = (A(x) \wedge B(x))$.
ii) $(A \cup B)(x) = sup\{A(x), B(x)\} = (A(x) \vee B(x))$.
iii) $(A.B)(x) = A(x)B(x)$.
iv) $(A + B)(x) = A(x) + B(x) - A(x)B(x)$.
v) $(A \oplus B)(x) = inf\{1, A(x) + B(x)\}$.
vi) $(A \ominus B)(x) = sup\{0, A(x) + B(x) - 1\}$.

Note 1.1. If $A \in f(R)$, then the set of $A_t = \{x \in R | A(x) \geq t\}$, $\forall t \in [0, 1]$ is called a "*level-set*" of A or a "$t - cut$" of A.

Theorem 1.1. *A is a fuzzy subdomain of R if and only if for all $t \in [0, 1]$ the t-level set of A_t is a domain.*

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* Department of Mathematics, Faculty of Science, University of Yasouj, Yasouj, IRAN
P.O.Box:75918-74934.

Proof. See Theorem "1.2" of [1]. □

The following theorem is an easy consequence of Definition "1.3" and Theorem "1.1".

Theorem 1.2. *If $A, B \in F(D)$ then:*
i) $(A \cap B) \in F(D)$
ii) $(AB) \in F(D)$
iii) $(A + B) \in F(D)$
iv) $(A \oplus B) \in F(D)$
v) $(A \ominus B) \in F(D)$.

Proof. See Theorem "1.3" of [1]. □

Theorem 1.3. *Let A, B and $C \in F(D)$, then $AoB \subseteq C$ if and only if $AB \subseteq C$.*

Proof. See Theorem "1.4" of [1]. □

Corollary 1.1. *For each $A, B \in F(D)$, we have:*

$$(A \ominus B) \subseteq (A.B) \subseteq (A \cap B) \subseteq (A + B) \subseteq (A \oplus B).$$

Corollary 1.2. *For any arbitrary family of fuzzy subdomains $\{D_\alpha\}_{\alpha\in\Lambda} \in F(D)$ we have:*

$$D = \bigcap_{\alpha\in\Lambda} D_\alpha \in F(D).$$

Where $\bigcap_{\alpha\in\Lambda} D_\alpha(x) = inf\{D_\alpha(X) | x \in D\}$.

Theorem 1.4. *Let $A, BC \in F(D)$ and $B(0) = C(0)$. Then*

$$A(B + C) = AB + AC$$

Proof. See Theorem "1.5" of [1]. □

Corollary 1.3. *If D is a commutative domain then for each $A, B \in F(D)$:*

$$(AB) = (BA)$$

Corollary 1.4. *For each $A, B, C \in F(D)$*

$$if \quad A \subseteq B \quad then \quad AC \subseteq BC$$

2. G-Fuzzy Structures

Definition 2.1. Let $A \in F(R)$, A is a left(right) fuzzy ideal of R if and only if $\forall a, b \in R$ we have:
i) $A(a, b) \geq inf\{A(a), A(b)\}$
ii) $A(ab) \geq A(b)(A(ab) \geq A(a))$.
A ia a fuzzy ideal of R if and only if A is a left fuzzy ideal and right fuzzy ideal of R.

Note 2.1. The condition of "i" in the Definition of "2.1" is equivalent to two following conditions:
i) $A(a + b) \geq inf\{A(a), A(b)\}, \quad \forall a, b \in R$
ii) $A(-a) = A(a), \quad \forall a \in A$.

Theorem 2.1. *For each $A \in F(R)$, A is a fuzzy ideal of R if and only if for every $t \in [0, 1]$, that $A_t \neq \emptyset$, A_t is an ideal of R.*

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Proof. Suppose that A is a fuzzy prime ideal of R, let $t \in [0,1]$ such that A_t is a proper ideal of R, let $a, b \in R$ such that $a \wedge b \in A_t$. Then $A(a \wedge b) \geq t$. Therefore $A(a \wedge b) \geq t \Rightarrow A(a) \vee A(b) \geq t$ (Since A is a fuzzy prime ideal). Thus $A(a) \geq t$ or $A(b) \geq t$. So $a \in A_t$ or $b \in A_t$. Using the fact that A_t is an ideal of R, we conclude that A_t is a prime ideal of R.
Conversely, suppose that for any $t \in [0,1]$ such that A_t is a proper ideal of R, A_t is a prime ideal of R. Let $x, y \in R$ and $t = A(x \wedge y)$. We have $x \wedge y \in A_t$. Hence $x \in A_t$ or $y \in A_t$(Since A_t is a prime ideal). It follows that $A(x) \geq t = A(x \wedge y)$ or $A(y) \geq t = A(x \wedge y)$, then $A(x \wedge y) \leq A(x) \vee A(y)$. Therefore, A is a fuzzy prime ideal of R. □

Remark 2.1. The important applications of the Theorem "2.1 " is the producing of fuzzy ideals of a ring.

Corollary 2.1. *Let $S \subseteq R$, χ_S (the Characteristic function on S) is a fuzzy ideal of R if and only if S is an ideal of R.*

Proof. This is obvious, because for any $t \in (0,1]$, the $(\chi_S)_t = S$ and $(\chi_S)_0 = R$. □

Note 2.2. i) The set of all fuzzy ideals on the ring of R is called by $I(R)$.
ii) If a fuzzy subset S of R define by $S(x) = sup\{t \in [0,1] | x \in I_t\}$ for all $x \in R$, then S is a fuzzy ideal of R.

Theorem 2.2. *Let $\{A_t | t \in [0,1]\}$ be a collection of ideals of R such that:*
i) $R = \bigcup_{t \in [0,1]} A_t$
ii) $t_1 \geq t_2 \Longleftrightarrow A_{t_1} \subseteq A_{t_2}, \qquad \forall t_1, t_2 \in [0,1]$.

Proof. It is sufficient to prove that S_t is an ideal of R, for every $t \in [0,1]$ with $S_t \neq \emptyset$.
Let $t \in [0,1]$. we have two cases: $1) t_1 = sup\{t \in [0,1] | t < t_1\}$,
$2) t_1 \neq sup\{t \in [0,1] | t < t_1\}$.

Case (1) implies that $x \in S_t \Longleftrightarrow x \in I_r, \forall r < t, r \in [0,1]$. i.e.,

$$x \in \bigcap_{\substack{r<t \\ r \in [0,1]}} I_r.$$

Hence

$$S_t = \bigcap_{\substack{r<t \\ r \in [0,1]}} I_r.$$

Which is an ideal of R.
For the case (2), there exists $\epsilon > 0$ such that $[t-\epsilon, t] \cap [0,1] = \emptyset$. If $x \in \bigcup_{\substack{r<t \\ r \in [0,1]}} I_r$, then $x \in S_r$ for some $r \geq t$. It follows that $S(x) \geq r \geq t$, so that $x \in S_t$. That is $x \in \bigcup_{\substack{r<t \\ r \in [0,1]}} I_r \subseteq S_t$.
Conversely, if $x \notin \bigcup_{\substack{r<t \\ r \in [0,1]}} I_r$, then $x \notin I_r$ for all $r \geq t$. Which implies that $x \notin I_r$ for all $r > t - \epsilon$, that is, if $xinI_r$ then $r \geq t - \epsilon$. Thus $S(x) \geq t - \epsilon$ and so $x \notin S_t$. Consequently

$$S_t = \bigcup_{\substack{r<t \\ r \in [0,1]}} I_r$$

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Which is an ideal of R.

□

Corollary 2.2. *Let $A \in I(R)$, for every $t_1, t_2 \in Im(A)$ the two level-sets A_{t_1} and A_{t_2} of A are equal if and only if there is no element $x \in R$ such that:*

$$t_1 < A(x) < t_2$$

Example 2.1. *Let $A : \mathbb{Z} \longrightarrow [0, 1]$ is defined as the following:*

$$A(x) = \begin{cases} 1 & x = 0 \\ 1/2 & x \in 4\mathbb{Z} - \{0\} \\ 1/3 & x \in 2\mathbb{Z} - 4\mathbb{Z} \\ 0 & x \in \mathbb{Z} - 2\mathbb{Z} \end{cases}$$

By the Note "1.1" the nonempty level-subsets of A are:
$A_0 = \mathbb{Z}$, $A_{\frac{1}{3}} = 2\mathbb{Z}$, $A_{\frac{1}{2}} = 4\mathbb{Z}$ and $A_1 =< 0 >$, since by the Theorem "2.1" A_i for every "i" is an ideal of $\mathbb{Z}$, then A is a fuzzy ideal of $\mathbb{Z}$.

Definition 2.2. Let $P \in I(R)$ be an arbitrary and nofix, P is a fuzzy prime ideal if and only if $\forall I, J \in I(R)$, $IJ \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$.

Corollary 2.3. *Let $I \in I(R)$, $t \in Im(I)$, the I_t is a prime ideal of R if:*

$$I(x) < I(y) \Longrightarrow I(xy) = I(y), \qquad \forall x, y \in R$$

Corollary 2.4. *If $P \in P(R)$ ($P(R)$:the set of all prime fuzzy ideals on R), then:*

$$P(xy) = max\{P(x), P(y)\} \qquad , \qquad \forall x, x \in R.$$

Remark 2.2 (12)**.** Let D be an integral domain with quotient field K, the following two statements are equivalent:
i) K is a finitely generated ring over D.
ii) K as a ring, can be generated over D by one element.

Definition 2.3. An integral domain satisfying either (hence both) of the statements in Remark "2.2" is called a $G-domain$.

The name honors Oscar Goldman. His paper [11] appeared at virtually the same time as a similar paper br Krull [13]. Since Krull already has a class of rings named after him, it seems advisable not to attempt to honor Ktull in this connection. $G-domains$ were also considered by Artin and Tate in [4]. Further results concerning the material in this section appear in Gilmer's paper [10].

Example 2.2. *1) Each field is a $G-domain$.*
2) A principal ideal domain is a $G-domain$ if and only if it has only a finite number of primes(up to units).
3) A Noetherian domain is a $G-domain$ if and only if it has only a finite number of non-zero prime ideals such that all of which are maximal.

Definition 2.4. Let D be a $G-domain$, A is a G- fuzzy subdomain of D if for all $x, y \in D$ we have:
i) A is a fuzzy subdomain of D(i.e. $A \in F(D)$).
ii) If K be the quotient field of D, then $K = D[u^{-1}]$, for some $0 \neq u \in D$.

Theorem 2.3. *Let A be a G-fuzzy subdomain with quotient fuzzy subfield K and let B be a fuzzy subring lying between A and K, then B is a G-fuzzy subdomain.*

Proof. If $K = A[u^{-1}]$ be the quoitent fuzzy subfield of A, then $K = B[u^{-1}]$ is also the quotient fuzzy subfield of B, therefore B is a G-fuzzy subdomain.

□

Theorem 2.4. *A is a G-fuzzy subdomain if and only if for each $t \in [0,1]$, A_t is a $G-$domain.*

Proof. See the proof of Theorem "2.1". □

Example 2.3. *Let $\mathbb{Q}$ be the Rational numbers, since for each prime number $2,3,5,7,...$ the extended fields $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{2},\sqrt{3}]$, $\mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}]$ and $\mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7}]$ are $G-$domains. If we define $A(x)$ as the following:*

$$A(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 1/2 & x \in \mathbb{Q}[\sqrt{2}] - \mathbb{Q} \\ 1/3 & x \in \mathbb{Q}[\sqrt{2},\sqrt{3}] - \mathbb{Q}[\sqrt{2}] \\ 1/5 & x \in \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}] - \mathbb{Q}[\sqrt{2},\sqrt{3}] \\ 1/7 & x \in \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7}] - \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}] \\ 0 & x \in \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7},..] = \mathbb{R} \end{cases}$$

Since for each $t \in [0,1]$, A_t is a $G-$domain as the follows:
$A_0 = \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7},..] = \mathbb{R}$, $A_{\frac{1}{7}} = \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7}]$, $A_{\frac{1}{5}} = \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}]$, $A_{\frac{1}{3}} = \mathbb{Q}[\sqrt{2},\sqrt{3}]$, $A_{\frac{1}{2}} = \mathbb{Q}[\sqrt{2}]$ and $A_1 = \mathbb{Q}$. Hence A is a G-fuzzy subdomain of $\mathbb{R}$.

Definition 2.5. A prime fuzzy ideal P in a commutative ring R is a G-fuzzy ideal if R/P is a G-fuzzy subdomain.

Theorem 2.5. *The fuzzy nilradical N of any commutative ring R is the intersection of all G-fuzzy ideals in R.*

Proof. Clearly a nilpotent element lies in every prime ideal. Conversly , suppose $u \notin N$. We must constract a G-fuzzy ideal excluding u. The fuzzy ideal 0 is disjoint from $\{u^n\}$. We can expand it to an fuzzy ideal P maximal repect to disjointness from $\{u^n\}$; We know that P is the fuzzy prime ideal.Now we show further that P is a G-fuzzy ideal. In the domain $R^* = R/P$, let u^* denote the image of u. The maximality of P tells us that every non-zero prime in R^* contains u^*. Therefore R^* is a G-fuzzy subdomain and P is also a G-fuzzy ideal. □

It is suggested that further research in this direction is likely going to reveal additional properties of G-type fuzzy ideals associated to fuzzy subdomains and thus contribute to our understanding of how such structures defines on the underlying G-type fuzzy subdomains.

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Author

Department of Mathematics, Faculty of Science, University of Yasouj, Yasouj, IRAN
E-mail address: aalizadeh@yu.ac.ir