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Matrix Game Under Uncertainty Theory via Entropy

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Abstract

The uncertainty is the new approach on the entities to obtain appropriate properties, when lack of information and knowledge about the entities. The uncertainty plays a major role in game theory and other human interactions to apply on real life problem. The developments have leads to the emergence of a new area called uncertain matrix game whose payoff elements are approximately known that represent as uncertain variables. Define the properties of uncertain matrix game, called as uncertain expected value operator and uncertain minimax equilibrium strategy on the uncertain matrix game. Introduce the entropy function on the strategy of the matrix game to formulate the new model. The theorem have been shown that there always exists at least an optimum solution of uncertain matrix game. Using uncertainty theory the uncertain matrix game converts into crisp linear programming problem depends upon the confidence level and solve it using genetic algorithm. The solution procedure have been discuss on both the uncertain variables, firstly consider the payoff elements are linear uncertain variables and then, zigzag uncertain variables. A numerical example have been illustrate to proposed the methodology.

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Keywords : Two-person zero-sum game, uncertain variable, linear uncertain variable, zigzag uncertain variable, genetic algorithm, uncertain expected value operator, uncertain minimax equilibrium strategy.

1 Introduction

Game theory [14],[15] is the theory of independent and interdependent decision making. It is concerned with decision making in organizations where the outcome depends on the decisions of two or more autonomous players, one of which may be natural itself, and where no signal decision maker has full control over the outcome. Some times the classical models fail to deal with interdependent decision making because the problems have been difficult to defined in the crisp scenario. So the developments has lead to the new area like fuzzy matrix [12] and bimatrix game, credibilistic matrix [13] and bimatrix [8] games, rough bimatrix [10] game, bifuzzy matrix [9] and bimatrix [11] game and so on. But main solution concepts is same as like crisp scenario. In all the uncertain types of game firstly, convert into crisp scenario, which depends upon the confidence level and solve it using traditional method. There have been various solutions which depends upon the confidence level and applier can find the optimal solution for appropriate confidence level.

The uncertainty is the mathematical tool to model imprecise quantities of the entities. Uncertainty theory was founded by B. Liu [3] in 2007. The first fundamental concept in uncertainty theory is uncertain measure that is used to measure the belief degree of an uncertain event. The concepts of membership function and uncertainty distribution are two basic tools to describe uncertain sets, where membership function is intuitionistic for us but frangible for arithmetic operations, and uncertainty distribution is hard-to-understand for us but easy-to-use for arithmetic operations. Fortunately, an uncertainty distribution may be uniquely determined by a membership function. The concept of uncertain variable (neither random variable nor fuzzy variable) in order to describe imprecise quantities in human systems. There are two type of uncertain variables, linear uncertain variable and zigzag uncertain variable defined by B. Liu [4].

It is natural to ask that, there can define matrix game where the payoff elements can not be crisp and its follow the properties of uncertainty theory. Entropy optimization models have been successfully applied to practical problems in many scientific and engineering disciplines. In this paper, we consider the matrix game whose payoff elements are only approximately known and can be represented by uncertain variable (linear uncertain and zigzag uncertain)and defined as uncertain matrix game. In the crisp scenario, there

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exists a beautiful relationship between two person zero sum matrix game and duality in linear programming [15]. It is therefore natural to ask if something similar holds in the uncertain scenario as well. This discussion essentially constitutes the core of our presentation. Firstly, describing the properties of uncertain theory. Then considering the game whose payoff elements are characterized as uncertain variables and its properties like uncertain equilibrium strategy, uncertain expected value operator and uncertain constraints, entropy function and its existence of solution. Finally, the uncertain matrix game converts into crisp mathematical model depends upon the confidence levels with dual each other. An example has been illustrated to validate our proposed methodology and presented conclusion of our paper.

2 Preliminaries

Let Γ be a nonempty set. A collection $\mathcal{L}$ of subsets of Γ is called a $\sigma-$ algebra if (a) $\Gamma \in \mathcal{L}$ (b) if $\Lambda \in \mathcal{L}$ then $\Lambda^c \in \mathcal{L}$ and (c) if $\Lambda_1, \Lambda_1, \cdots \in \mathcal{L}$ then $\Lambda_1 \cup \Lambda_1 \cup \cdots \in \mathcal{L}$ Each element Λ in the $\sigma-$algebra $\mathcal{L}$ is called an event. Uncertain measure is a function from $\mathcal{L}$ to $[0, 1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $\mathcal{M}\{\Lambda\}$ which indicates the belief degree that Λ will occur. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, B. Liu [3] proposed the following four axioms:
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (Monotonicity Axiom) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$.
Axiom 3. (Self-Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 4. (Countable Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have,

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

Definition 2.1 *(B. Liu [3]) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, monotonicity, self-duality, and countable subadditivity axioms.*

$$0 \leq \mathcal{M}\{\Lambda\} \leq 1 \quad \textit{for any event} \quad \Lambda$$
$$\mathcal{M}\{\varnothing\} = 0 \quad \textit{where } \varnothing \quad \textit{is a empty set.}$$

Definition 2.2 *(B. Liu [3]) Let Γ be a nonempty set, $\pounds$ a $\sigma-$ algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \pounds, \mathcal{M})$ is called an uncertainty space.*

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Definition 2.3 *(B. Liu [3]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e, for any Borel set $\mathbb{B}$ of real numbers, the set*

$$\{\xi \in \mathbb{B}\} = \{\gamma \in \Gamma | \xi(\gamma) \in \mathbb{B}\}$$

Definition 2.4 *(B. Liu [3]) The uncertainty distribution $\Phi : \Re \to [0, 1]$ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition 2.5 *(B. Liu [4]) An uncertain variable ξ is called linear if it has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x \geq b \end{cases}$$

denoted by $\mathcal{L}(a, b)$ where a and b are real numbers with $a < b$. Then for any $\alpha \in (0, 1]$

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b$$

Definition 2.6 *(B. Liu [4]) An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution*

$$\Phi(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)} & \text{if } b \leq x \leq c \\ 1 & \text{if } x \geq c \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b and c are real numbers with $a < b < c$. Then for any $\alpha \in (0, 1]$

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c & \text{if } \alpha \geq 0.5 \end{cases}$$

Theorem 2.1 *(B. Liu [4]) Assume that ξ_1 and ξ_2 are independent linear uncertain variables $\mathcal{L}(a_1, b_1)$ and $\mathcal{L}(a_2, b_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$, i.e,*

$$\mathcal{L}(a_1, b_1) + \mathcal{L}(a_2, b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2)$$

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The product of a linear uncertain variable $\mathcal{L}(a,b)$ *and a scalar number* $k > 0$ *is also a linear uncertain variable* $\mathcal{L}(ka,kb)$, *i.e,*

$$k\mathcal{L}(a,b) = \mathcal{L}(ka,kb)$$

Theorem 2.2 *(B. Liu [4]) Assume that* ξ_1 *and* ξ_2 *are independent zigzag uncertain variables* $\mathcal{Z}(a_1,b_1,c_1)$ *and* $\mathcal{Z}(a_2,b_2,c_2)$, *respectively. Then the sum* $\xi_1+\xi_2$ *is also a zigzag uncertain variable* $\mathcal{Z}(a_1+a_2,b_1+b_2,c_1,c_2)$, *i.e,*

$$\mathcal{Z}(a_1,b_1,c_1) + \mathcal{L}(a_2,b_2,c_2) = \mathcal{L}(a_1+a_2,b_1+b_2,c_1+c_2)$$

The product of a linear uncertain variable $\mathcal{Z}(a,b,c)$ *and a scalar number* $k > 0$ *is also a linear uncertain variable* $\mathcal{L}(ka,kb,kc)$, *i.e,*

$$k\mathcal{Z}(a,b,c) = \mathcal{Z}(ka,kb,kc)$$

Definition 2.7 *(B. Liu [4]) Let* ξ *be an uncertain variable. Then the expected value of* ξ *is defined by*

$$\begin{aligned} E[\xi] &= \int_0^{\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr \\ E[\xi] &= \int_0^{\infty} (1-\Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx \end{aligned}$$

where Φ *is the uncertainty distribution function of* ξ *, provided that at least one of the two integrals is finite.*
If $\xi \in \mathcal{L}(a,b)$ *be a linear uncertain variable, then the expected value is* $E[\xi] = \frac{a+b}{2}$.
If $\xi \in \mathcal{Z}(a,b,c)$ *be a zigzag uncertain variable, then the expected value of* ξ *is* $E[\xi] = \frac{a+2b+c}{4}$
Let ξ *and* η *be independent uncertain variables with finite expected values. Then for any real numbers* a *and* b, *we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

Definition 2.8 *(B. Liu [4]) Let* ξ *be an uncertain variable with uncertainty distribution* Φ, *and* $\alpha \in (0,1]$. *Then*

$$\begin{aligned} \xi_{\sup}(\alpha) &= \sup\{r | \mathcal{M}\{\xi \geq r\} \geq \alpha\} \\ \xi_{\sup}(\alpha) &= \Phi^{-1}(1-\alpha) \end{aligned}$$

is called the α*-optimistic value to* ξ, *and*

$$\begin{aligned} \xi_{\inf}(\alpha) &= \inf\{r | \mathcal{M}\{\xi \leq r\} \geq \alpha\} \\ \xi_{\inf}(\alpha) &= \Phi^{-1}(\alpha) \end{aligned}$$

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is called the α-pessimistic value to ξ.
Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$. Then its $\alpha-$ optimistic and $\alpha-$pessimistic values are

$$\begin{aligned} \xi_{\sup}(\alpha) &= \alpha a + (1-\alpha)b \\ \xi_{\inf}(\alpha) &= (1-\alpha)a + \alpha b \end{aligned}$$

Let ξ be a zigzag uncertain variable $\mathcal{Z}(a,b,c)$. Then its $\alpha-$optimistic and $\alpha-$pessimistic values are

$$\xi_{\sup}(\alpha) = \begin{cases} 2\alpha b + (1-2\alpha)c & \text{if } \alpha < 0.5 \\ (2\alpha-1)a + (2-2\alpha)b & \text{if } \alpha \geq 0.5 \end{cases}$$

$$\xi_{\inf}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c & \text{if } \alpha \geq 0.5 \end{cases}$$

Definition 2.9 *(B. Liu [4]) Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{\infty} S(\Phi(x))dx$$

where $S(t) = -t\ln(t) - (1-t)\ln(1-t)$
Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$. Then its entropy is

$$H[\xi] = -\int_a^b \left(\frac{x-a}{b-a} \ln \frac{x-a}{b-a} + \frac{b-x}{b-a} \ln \frac{b-x}{b-a} \right) dx = \frac{b-a}{2}$$

Let ξ be a zigzag uncertain variable $\mathcal{Z}(a,b,c)$. Then its entropy is

$$H[\xi] = \frac{c-a}{2}$$

Theorem 2.3 *(B. Liu [4]) Assume that $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n$ are independently linear uncertain variables $\mathcal{L}(a_1,b_1), \mathcal{L}(a_2,b_2), \cdots, \mathcal{L}(a_n,b_n)$ respectively. Then for any confidence level $\alpha \in (0,1)$, the uncertain constraint*

$$\mathcal{M}\left\{ \sum_{i=1}^{n} \xi_i x_i \leq 0 \right\} \geq \alpha$$

holds if and only if

$$\sum_{i=1}^{n} ((1-\alpha)a_i + \alpha b_i)x_i \leq 0$$

Assume that $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n$ are independently zigzag uncertain variables $\mathcal{Z}(a_1, b_1, c_1), \mathcal{Z}(a_2, b_2, c_2), \cdots, \mathcal{Z}(a_n, b_n, c_n)$ respectively. Then for any confidence level $\alpha \in (0, 1)$, the uncertain constraint

$$\mathcal{M}\left\{\sum_{i=1}^{n} \xi_i x_i \leq 0\right\} \geq \alpha$$

holds if and only if,
when $\alpha < 0.5$

$$\sum_{i=1}^{n}((1-2\alpha)a_i + 2\alpha b_i)x_i \leq 0$$

when $\alpha \geq 0.5$

$$\sum_{i=1}^{n}((2-2\alpha)b_i + (2\alpha-1)c_i)x_i \leq 0$$

3 Uncertain Matrix Game

In two-person zero-sum game, both players have same pay-off matrix. Suppose the elements of the pay-off matrix are not defined properly due to incomplete and imprecise information or lack of knowledge about the real-life data on matrix game. In traditional point of view, considering the pay-off elements are fuzzy or random variables and solve the matrix game. But, sometimes the practical point of view, fuzziness or randomness are not satisfied for the pay-off elements in matrix game. In this situation, the elements may be considered as uncertain variable and defined as uncertain matrix game for two-person zero-sum matrix game.

Let $X \equiv \{1, 2, \cdots, m\}$ and $Y \equiv \{1, 2, \cdots, n\}$ be the set of strategies for players I and II respectively. Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Let $e^T = \{1, 1, \cdots, 1\}$ be the vector of elements '1' whose dimension is specified as per specific context. Mixed strategies of players I and II are represented by weights to their strategies $S^X = \{x \in R^m_+, e^T x = 1\}$ and $S^Y = \{y \in R^n_+, e^T y = 1\}$ respectively.

We consider the uncertain variable ξ_{ij} associated with pay-off element which implies that the player I gains or player II loses when player I played the strategy i and player II played the strategy j. Then the two-person zero-sum matrix game is represented by

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uncertain pay-off matrix as follows :

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} & . & . & . & \xi_{1n} \\ \xi_{21} & \xi_{22} & . & . & . & \xi_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \xi_{m1} & \xi_{m2} & . & . & . & \xi_{mn} \end{pmatrix}$$

The meaning of the solution of the uncertain matrix game $G^{\xi} = \{S^X, S^Y, \xi\}$ is best understood in terms of maxmin and minmax principle for players I and II respectively. Then the maxminimizer of player I is the solution of the uncertain expected value v^* i.e,

$$v^* = \max_{x \in S^X} \min_{y \in S^Y} E[x^T \xi y]$$

and the minmaximizer of player II is the solution of the uncertain expected value w^* where,

$$w^* = \min_{y \in S^Y} \max_{x \in S^X} E[x^T \xi y]$$

Definition 3.1 *Let $\xi = (\xi_{ij})_{m \times n}$ $(i = 1, 2, ..., m, j = 1, 2, ..., n)$ be uncertain variables with finite expected values. Then $(x^*, y^*) \in S^X \times S^Y$ is called a uncertain expected minimax equilibrium strategy to the uncertain matrix game $G^{\xi} = \{S^X, S^Y, \xi\}$ if,*

$$E[x^T \xi y^*] \leq E[{x^*}^T \xi y^*] \leq E[{x^*}^T \xi y]$$

Assuming that player I's optimal decision criteria is to maximize the critical value of his uncertain payoff $x^T \xi y$ at given confidence level $\alpha \in (0, 1]$. Then a maxminimizer of player I is the solution of the uncertain constrained programming as follows:

$$\max_{x \in S^X} \min_{y \in S^Y} \max_{v} \mathcal{M}\{x^T \varrho y \geq v\} \geq \alpha$$

Definition 3.2 *Let $\xi_{ij} (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ be independent uncertain variables, $\alpha \in (0, 1]$ and $v \in R$ be the predetermined level of the uncertain pay-offs. Then (x^*, y^*) is called a α-uncertain equilibrium to the uncertain matrix game $G^{\varrho} = \{S^X, S^Y, \varrho\}$ if,*

$$\max\{v |\ \mathcal{M}\{x^T \xi y^* \geq v\} \geq \alpha\} \leq \max\{v |\ \mathcal{M}\{{x^*}^T \xi y^* \geq v\} \geq \alpha\} \leq \max\{v |\ \mathcal{M}\{{x^*}^T \xi y \geq v\} \geq \alpha\}$$

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Theorem 3.1 *In a two-person zero-sum game, the uncertain payoffs $\xi = (\xi_{ij})_{m\times n}$ $(i = 1,2,...,m, j = 1,2,...,n)$ be characterized as uncertain variables with finite expected values. Then there exists at least a uncertain expected minimax equilibrium strategy to the uncertain matrix game $G^{\xi} = \{S^X, S^Y, \xi\}$.*

Proof : For any $y \in S^Y$, let

$$Q(y) = \{\overline{x} \in S^X : \ E[x^T\xi y] \le E[\overline{x}^T\xi y], \ \forall x \in S^X\}$$

then $Q(y) \subset S^X$ For any $x \in S^X$, let

$$P(x) = \{\overline{y} \in S^Y : \ E[x^T\xi\overline{y}] \le E[x^T\xi y], \ \forall y \in S^Y\}$$

then $P(x) \subset S^Y$. We first prove that $Q(y)$ and $P(x)$ are both convex sets. For any $x_1, x_2 \in Q(y)$, it is clear that $\lambda x_1 + (1-\lambda)x_2 \in S^X$ with $\lambda \in [0,1]$. Since the components of x_1, x_2 are all nonnegative real numbers, it follows that,

$$E[(\lambda x_1 + (1-\lambda)x_2)^T\xi y] = \lambda E[{x_1}^T\xi y] + (1-\lambda)E[{x_2}^T\xi y]$$

Moreover, it follows from above that $E[{x_1}^T\xi y] \ge E[x^T\xi y]$ and $E[{x_2}^T\xi y] \ge E[x^T\xi y]$ for any $x \in S^X$. Thus for any $\lambda \in [0,1]$, $\lambda E[{x_1}^T\xi y] \ge \lambda E[x^T\xi y]$ and $(1-\lambda)E[{x_2}^T\xi y] \ge (1-\lambda)E[x^T\xi y]$. Hence, we have,

$$E[(\lambda x_1 + (1-\lambda)x_2)^T\xi y] \ge E[x^T\xi y], \ \forall x \in S^X$$

This implies that

$$\lambda x_1 + (1-\lambda)x_2 \in Q(y)$$

Hence $Q(y)$ is a convex set. Similarly, we can prove that $P(x)$ is a convex set. Let $F : S^X \times S^Y \to P(S^X \times S^Y)$ be a set-valued mapping defined by

$$F(z) = Q(y) \times P(x), \ \forall z = (x^T, y^T) \in S^X \times S^Y,$$

where $P(S^X \times S^Y)$ is the power set of $S^X \times S^Y$. Then we have,

$$(u,v) \in F(z) \Leftrightarrow u \in Q(y), v \in P(x)$$

Let $z_n = ({x_n}^T, {y_n}^T)^T$ and $(u_n, v_n) \in F(z_n)$, where $u_n \to u_0, v_n \to v_0, x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$. Since $u_n \in Q(y_n)$, for any $x \in S^X$, we obtain

$$E[x^T\xi y_n] \le E[{u_n}^T\xi y_n]$$

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Hence,

$$\lim_{n\to+\infty} E[x^T \xi y_n] \leq \lim_{n\to+\infty} E[{u_n}^T \xi y_n]$$

It implies that

$$E[x^T \xi y_0] \leq E[{u_0}^T \xi y_0]$$

Thus $u_0 \in Q(y_0)$. Similarly, since $v_n \in P(x_n)$, for any $y \in S^Y$, we have,

$$E[{x_n}^T \xi v_n] \leq E[{x_n}^T \xi y]$$

Hence,

$$\lim_{n\to+\infty} E[{x_n}^T \xi v_n] \leq \lim_{n\to+\infty} E[{x_n}^T \xi y]$$

It implies that

$$E[{x_0}^T \xi v_0] \leq E[{x_0}^T \xi y]$$

Thus $v_0 \in P(x_0)$. Hence $Q(y)$ and $P(x)$ are both convex closed sets and the graph of F is convex closed. It is clear that the set-valued maping F is upper semi-continuous with non-empty, convex closed values. It follows from Kakutani's fixed-point theorem that there exists at least a point $z^* \in S^X \times S^Y$ such that $z^* \in F(z^*)$, i.e, there exists at least a point $(x^*, y^*) \in (S^X, S^Y)$, such that

$$x^* \in Q(y^*) \Rightarrow E[x^T \xi y^*] \leq E[{x^*}^T \xi y^*],\ \forall x \in S^X$$
$$y^* \in P(x^*) \Rightarrow E[{x^*}^T \xi y^*] \leq E[{x^*}^T \xi y],\ \forall y \in S^Y$$

The theorem is proved.
If for each uncertain variables ξ_{ij}, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ are independent, then for any mixed strategies x and y, it follows from the properties that

$$E[x^T \xi y] = E[\sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_j] = \sum_{j=1}^{n} \sum_{i=1}^{m} E[\xi_{ij}] x_i y_j$$

It is well known that every two-person zero-sum matrix game with crisp payoffs has a minimax equilibrium strategy, then the matrix game with uncertain payoffs has a uncertain minimax equilibrium strategy.

4 Linear Programming Problem (LPP)

The LPP is one of the most widely used techniques in operations research, it is defined as means of maximizing a quantity known as objective function, subject to a set of

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constraints represented by equations and inequalities. It is very well known that every two-person zero-sum matrix game is equivalent to two linear programming problems which are dual to each other. If any one of these two mutually dual linear programming problems is solved, then the solution of the other can be made by using linear duality theory [14].

Let us consider the uncertain matrix game $G^{\xi} = \{S^X, S^Y, \xi\}$ then for the player II problem is

$$\min_{y \in S^Y} \max_{x \in S^X} x^T \xi y$$

where Min and Max operators are defined as

$$\min_{y \in S^Y} x^T \xi y = x^T \xi y_{\min} \text{ such that } E[x^T \xi y] \geq E[x^T \xi y_{\min}]$$

and

$$\max_{y \in S^Y} x^T \xi y = x^T \xi y_{\max} \text{ such that } E[x^T \xi y] \leq E[x^T \xi y_{\max}]$$

Since S^X and S^Y are compact convex sets and for a given y, $x^T \xi y$ is a linear function of x, then,

$$\max_{x \in S^X} x^T \xi y$$

will be attained at an extreme point of S^X. Therefore, for given $y \in S^Y$,

$$\max_{x \in S^X} x^T \xi y = \max_{1 \leq i \leq m} \sum_{j=1}^{n} \xi_{ij}\, y_j$$

If we consider,

$$w = \max_{1 \leq i \leq m} \sum_{j=1}^{n} \xi_{ij} y_j$$

Then the minmax value of the player II is obtained by solving the following linear programming problem with uncertain constraints as :

Model 1

$$\begin{aligned} & \min \; w \\ \text{subject to, } \sum_{j=1}^{n} \xi_{ij}\, y_j \; & \leq \; w \text{ for } i = 1, 2, \cdots, m \\ e^T y \; & = \; 1 \\ y \; & \geq \; 0 \end{aligned}$$

Applying the transformation $y_j^{'} = \frac{y_j}{w}$, then $e^T y^{'} = \frac{1}{w}$ and the above **Model 1** becomes into a standard linear programming problem with uncertain constraints as follows:
Model 2

$$\begin{aligned} & \max\ e^T y^{'} \\ \text{subject to,}\quad & \sum_{j=1}^{n} \xi_{ij}\ y_j^{'} \ \le\ 1 \ \text{ for }\ i = 1, 2, \cdots, m \qquad (4.1) \\ & y^{'} \ \ge\ 0 \end{aligned}$$

Applying the transformation $y_j^{'} = \frac{y_j}{w}$, then $e^T y^{'} = \frac{1}{w}$ and the above **Model 1** becomes into a standard linear programming problem with uncertain constraints as follows:
The entropy uncertain game model(Das and Roy) for both the players are given in the following models **Model 1e**. Then the LPP program becomes
Model 2e

$$\begin{aligned} & \max\ e^T y^{'} \\ & \max\ entr1 \qquad (4.2) \\ \text{subject to,}\quad & \sum_{j=1}^{n} \xi_{ij}\ y_j^{'} \ \le\ 1 \ \text{ for }\ i = 1, 2, \cdots, m \qquad (4.3) \\ & entr1 = -\sum_{j=1}^{n} y_j^{'} \ln(y_j^{'}) \qquad (4.4) \\ & y^{'} \ \ge\ 0 \end{aligned}$$

Again, the constraints (4.1) are not deterministic feasible region due to present of uncertain variables. A natural idea is to provide the confidence levels $\alpha \in (0, 1]$, of the uncertain measure $\mathcal{M}$ at which it desires that the uncertain constraints hold. Thus we have,

$$\mathcal{M}\{\sum_{j=1}^{n} \xi_{ij}\ y_j^{'} - 1 \le 0\} \ge \alpha \quad \text{for} \quad i = 1, 2, \cdots, m$$

Using above real constraints into the **Theorem 2.3**, we can find the crisp constraints for $i = 1, 2, \cdots, m$ from the uncertain constraints (4.1). Let us assume that the uncertain payoff matrix $\xi_{ij} = \mathcal{L}(a_{ij}, b_{ij})$ be the linear uncertain variables Then, the **Model 2** with real constraints can be defined, depending upon the confidence level as follows:

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Model 3

$$
\begin{aligned}
\max \ e^T y^{'} \\
\max \ entr1 \\
\text{subject to,} \quad \sum_{j=1}^{n}((1-\alpha)a_{ij}+\alpha b_{ij})\ y_j^{'} &\leq 1 \ \text{ for } \ i=1,2,\cdots,m \\
entr1 = -\sum_{j=1}^{n} y_j^{'} \ln(y_j^{'}) \\
y^{'} &\geq 0
\end{aligned}
$$

Let us assume that the uncertain payoff matrix $\xi_{ij} = \mathcal{Z}(a_{ij}, b_{ij}, c_{ij})$ be the zigzag uncertain variables Then, the **Model 2** with real constraints can be defined, depending upon the confidence level as follows:

Model 4

$$
\begin{aligned}
\max \ e^T y^{'} \\
\max \ entr1 \\
\text{subject to,} \\
\text{if } \alpha < 0.5, \\
\sum_{j=1}^{n}((1-2\alpha)a_{ij}+2\alpha b_{ij})\ y_j^{'} &\leq 1 \ \text{ for } \ i=1,2,\cdots,m \\
\text{if } \alpha \geq 0.5, \\
\sum_{j=1}^{n}((2-2\alpha)b_{ij}+(2\alpha-1)c_{ij})\ y_j^{'} &\leq 1 \ \text{ for } \ i=1,2,\cdots,m \\
entr1 = -\sum_{j=1}^{n} y_j^{'} \ln(y_j^{'}) \\
y^{'} &\geq 0
\end{aligned}
$$

For the equally weight of the objective function of entropy and strategy the model becomes

Model 3e

$$
\begin{aligned}
\max\ e^T y' &+ entr1 \\
\text{subject to,} \quad \sum_{j=1}^{n}((1-\alpha)a_{ij}+\alpha b_{ij})\ y'_j &\leq 1 \ \text{ for } \ i=1,2,\cdots,m \\
entr1 = -\sum_{j=1}^{n} y'_j \ln(y'_j) & \\
y' &\geq 0
\end{aligned}
$$

Model 4e

$$
\max\ e^T y' + entr1
$$

subject to,

if $\alpha < 0.5$,

$$
\sum_{j=1}^{n}((1-2\alpha)a_{ij}+2\alpha b_{ij})\ y'_j \leq 1 \ \text{ for } \ i=1,2,\cdots,m
$$

if $\alpha \geq 0.5$,

$$
\begin{aligned}
\sum_{j=1}^{n}((2-2\alpha)b_{ij}+(2\alpha-1)c_{ij})\ y'_j &\leq 1 \ \text{ for } \ i=1,2,\cdots,m \\
entr1 = -\sum_{j=1}^{n} y'_j \ln(y'_j) & \\
y' &\geq 0
\end{aligned}
$$

5 Genetic Algorithm(GA)

In this section, we introduce the genetic algorithm [5] to find the uncertain expected minimax equlibrium strategies and uncertain expected value of the game (optimal strategy and value of the game for the both players) of the uncertain matrix game. The chromosomes of the genetic algorithm are the strategies of the uncertain matrix game. The following steps are followed to find the optimal strategy and value of the game for the both players of the uncertain matrix game.

Initialization: Randomly selects the Population Size as 50 chromosome between $(0, 1)$ which satisfy the constraints of the uncertain matrix game.

Evaluation: Compute the objective function for all the chromosomes.

Selection: Select the particular chromosome which gives the optimum solution of the objection function of the uncertain matrix game.

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Crossover: Generate the new chromosome using crossover of the pair of chromosomes and the new chromosome must satisfy the constraints of the uncertain matrix game.
Mutation: Mutation is a genetic operator used to maintain genetic diversity from one generation of a population of genetic algorithm chromosomes to the next. Mutation alters one or more gene values in a chromosome from its initial state. In mutation, the solution may change entirely from the previous solution. Hence GA can come to better solution by using mutation.
Step 1: Initialization
Step 2: Evaluation for iteration 1
Step 3: Selection
Step 4: Crossover
Step 5: Mutation
Step 6: Evaluation for the current iteration
Step 7: At the end of iteration (500), we obtain the optimal solution otherwise goto **Step 3**.

6 Numerical Example

In order to show the applicability of the proposed methodology, let us assume that the linear uncertain payoff matrix is.

$$\xi = \begin{bmatrix} \mathcal{L}(180,190) & \mathcal{L}(156,158) \\ \mathcal{L}(90,95) & \mathcal{L}(180,190) \end{bmatrix}$$

then the player II LPP problem **Model 3e** is

$$\begin{aligned} \max \quad & y_1^{'} + y_2^{'} + entr1 \\ \text{subject to,} \quad & \\ & \{180(1-\alpha)+190\alpha\}\, y_1^{'} + \{156(1-\alpha)+158\alpha\}\, y_2^{'} \leq 1 \\ & \{90(1-\alpha)+95\alpha\}\, y_1^{'} + \{180(1-\alpha)+190\alpha\}\, y_2^{'} \leq 1 \\ & entr1 = -y_1^{'}\ln(y_1^{'}) - y_2^{'}\ln(y_2^{'}) \\ & y_1^{'}, y_2^{'} \geq 0 \end{aligned}$$

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For $\alpha = 0.5$, the above LPP can be written as

$$\begin{aligned} \max \quad y_1' + y_2' \quad &+ \quad entr1 \\ \text{subject to,} & \\ 185\ y_1' + 157\ y_2' \quad &\leq \quad 1 \\ 92.5\ y_1' + 185\ y_2' \quad &\leq \quad 1 \\ entr1 = -y_1' \ln(y_1') - y_2' \ln(y_2') & \\ y_1', y_2' \quad &\geq \quad 0 \end{aligned}$$

The solution of the above problem using GA is $y' = (0.001421139, 0.004694836)$ with strategy for the player II is $(0.232365, 0.767635)$ and value of the game is $w^8 = 163.506228$. If we consider the uncertainty payoff elements are zigzag uncertain variables

$$\xi = \begin{bmatrix} \mathcal{Z}(180,190,200) & \mathcal{Z}(156,158,160) \\ \mathcal{Z}(90,95,100) & \mathcal{Z}(180,190,200) \end{bmatrix}$$

For $\alpha = 0.2$, the the player II LPP problem **Model 4e** can be written as

$$\begin{aligned} \max \quad y_1' + y_2' \quad &+ \quad entr1 \\ \text{subject to,} & \\ 184\ y_1' + 156\ y_2' \quad &\leq \quad 1 \\ 92\ y_1' + 184\ y_2' \quad &\leq \quad 1 \\ entr1 = -y_1' \ln(y_1') - y_2' \ln(y_2') & \\ y_1', y_2' \quad &\geq \quad 0 \end{aligned}$$

The solution of the above problem using GA is $y^* = (0.001435603, 0.004716981)$ with strategy for the player II is $(0.233333, 0.766667)$ and value of the game is $w^* = 162.533336$. For $\alpha = 0.7$, the above LPP problem **Model 4e** can be written as

$$\begin{aligned} \max \quad y_1' + y_2' \quad &+ \quad entr1 \\ \text{subject to,} & \\ 194\ y_1' + 158\ y_2' \quad &\leq \quad 1 \\ 97\ y_1' + 194\ y_2' \quad &\leq \quad 1 \\ entr1 = -y_1' \ln(y_1') - y_2' \ln(y_2') & \\ y_1', y_2' \quad &\geq \quad 0 \end{aligned}$$

The solution of the above problem using GA is $y^* = (0.001613626, 0.004347826)$ with strategy for the player II is $(0.270677, 0.729323)$ and value of the game is $w^* = 167.744368$.

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7 Conclusion

Game theory provides a basic computational framework for formulating and analyzing the problem that the decision of one person depends on the decision of his opponents. Of the different types of game, the two person zero sum game, which is also called matrix game, with uncertainty payoff have been widely discussed. Some solution method on uncertain matrix game have been devised. In real world, there are cases in which the payoffs are not exactly known. This papers uses uncertainty variable to represent the uncertain data and developed a solution method of matrix game. Finally, the optimal strategy and the value of the game for the players of the proposed model developed with uncertain constraints (linear uncertain and zigzag uncertain) through applying GA by the uncertain measure with confidence level(chosen by decision maker). A numerical example demonstrates the feasibility of applying the uncertain programming approach to rough matrix game

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