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Mathematical Modeling and Numerical Simulation of Blood Flow through Tapered Artery

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Abstract: Solution of blood flow problem in a tapered artery is presented in this paper. Where blood is taken as non – Newtonian Casson's fluid. The effect of various parameter of blood flow with respect to tapered angle have been studied. Analytical expression for velocity, wall shear stress and volumetric flow rate is given in this paper. The graphical representations have been made to validate the analytical findings with a view of its applicability to stenotic diseases. It is found that wall shear stress increases when peak is obtained then decreases for different values of tapering angle.

Key words- Tapered artery**,** non-Newtonian fluid, Multiple stenosis, Blood flow, Shear stress.

Introduction: Cardiovascular diseases, particularly arthrosclerosis are the main cause for deaths. Arthrosclerosis is due to fatty substance3 deposits inside the arterial walls, which narrows the arteries. Various treatments are available to cure stenosis, like medication, bypass surgery, catheterization etc. catheterization is the simple and frequently used approach as the procedure can open the narrowed heart valves, blocked arteries and repair the defects[Biswas and Chakraborty 2009]. Many researchers found that the study of blood flow through tapered tubes is important not only for an understanding of the behavior of the marvelous body fluid in arteries, but also for design of prosthetic blood vessels [How and Black, 1987]. Akbar and Nadeem (2014) discussed heat and mass transfer effects on carreau fluid model for blood flow through a tapered artery with a stenosis. Arora (2011) studied the artificial neural network modeling for the system of blood flow through tapered artery with mild stenosis. Hazarika and Sharma (2014) have taken two layered mathematical model for blood flow through tapering asymmetric stenosed artery with slip velocity at a interface under the effect of transverse magnetic field. Many researchers studied the flow of blood as Newtonian and non- Newtonian fluid through tapered tubes. In this study pressure drop, pressure gradient and flux were measured in rigid wall model of tapered graphs under steady flow conditions. Both Newtonian and non-Newtonian fluids were examined. The results are compared with experimental data and theoretical predictions pressure drop, flow rate were observed and all four test sections for both Newtonian and non-Newtonian fluids. Mandal (2005) discussed about an unsteady analysis of non-Newtonian blood flow through tapered arteries with a stenosis. Kumar (2006) worked on the numerical study of axisymmetric blood flow in constricted rigid tube. Srikanth (2015) also find the shear stress distribution at the wall of omega shaped stenotic tapered artery in the presence of catheter. Due to above discussion no one can consider the time dependent geometry of stenosis through tapered artery, therefore we have considered the time dependent geometry of stenosis in tapered artery. In our blood is treated as casson fluid model. Velocity, wall shear stress and flow rate are obtained through graphs. The numerical values are calculated with Matlab and the results are compared with theoretical and experimental data. in this research paper we found that the velocity of the blood flow decreases with radius and also for different values of tapering angle.

Formulation of the problem:

The tapered blood vessel segment having a stenosis in its lumen is modelled as a thin elastic tube with a circular cross-section containing an incompressible non-Newtonian fluid characterised by casson fluid model. Let (r, z, φ) be the coordinates of a material point in the cylindrical polar coordinates systemwhere the z-axis is taken along the axis of the artery while r, φ are taken along the radial and the circumferential directions, respectively. The geometry of the time-variant stenosed arterial segment [Fig.(1)].

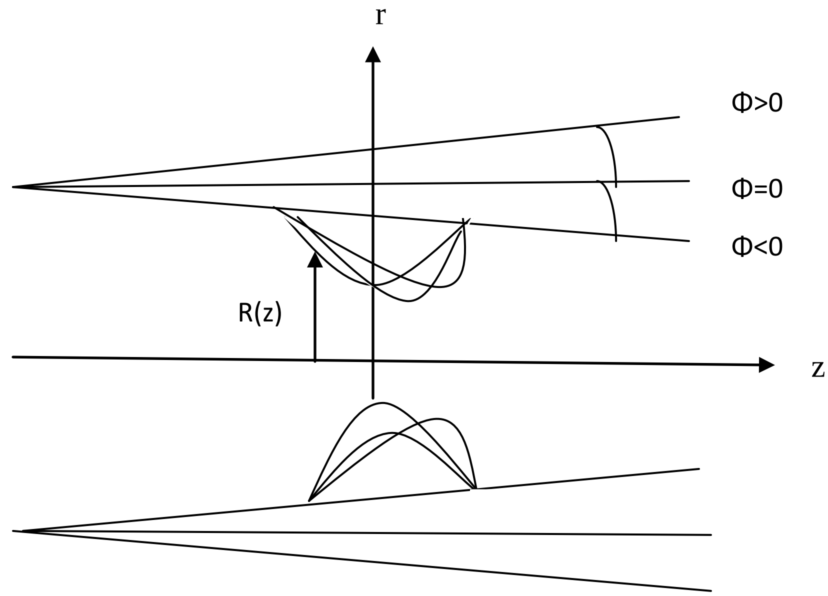

Fig.(1). Schematic diagram of blood flow in a stenosed tapered artery

Consider an axially symmetric, laminar, and fully developed flow of blood in the z direction. It can be shown that the radial velocity is negligibly small in its magnitude and may be neglated for a low mean Reynold number flow problem with mild stenosis. The momentum equation are

$$\rho\frac{\partial u^*}{\partial t^*} = -\frac{\partial p^*}{\partial z^*} - \frac{1}{r^*}\frac{\partial (r^* {\tau^*}_{rz})}{\partial r^*} \tag{1}$$

where for a casson fluid ${\tau^*}_{rz}$ is given by

$$|{\tau^*}_{rz}|^{1/2} = {\tau_y^*}^{1/2} + \sqrt{\mu}\left|\frac{\partial u^*}{\partial r^*}\right|^{1./2}, \text{ if } |{\tau^*}_{rz}| \geq \tau_y^*, \tag{2}$$

$$\frac{\partial u^*}{\partial r^*} = 0, \text{ if } |{\tau^*}_{rz}| \leq \tau_y^* \tag{3}$$

The boundary condition are

$u^* = 0$ at $r^* = R^*(z^*)$,, τ^* is finite at $r^* = 0$.t

Let us consider the pulsetile laminar flow of blood in the z direction through a compliant tube whose radius varies as (using non dimensional scheme)

$$R(z,t) = \begin{cases} (1-nz)(1-A_1(t)\delta_s\left[L_0^{(m-1)}(z-d)-(z-d)^m\right]), & if\ d \leq z \leq d+L_0, \\ 1, & \text{otherwise} \end{cases} \tag{4}$$

$$A_1(t) = \frac{[1 - e^{(-t/T)}]^{m\frac{m}{m-1}}}{aL_0^m(m-1)}, \quad m \neq 1$$

$\delta_s = hsec\varphi,$

$z = \frac{kd+(k-1)L_0+L_0/m^{1/(m-1)}}{\alpha}.$

To solve the above system of equations following non-dimensional variables are introduced:

$$u = \frac{u^*}{\frac{q_0 a^2}{4\mu}}, \tau = \frac{2\tau^*}{q_0 a}, r = \frac{r^*}{a}, z = \frac{z^*}{a}, t = t^*\omega, d = \frac{d^*}{a}, L = \frac{L^*}{a}, \delta = \frac{\delta^*}{a} \quad (5)$$

where q_0 is the constant pressure gradient.

The pressure gradient which is function of z^* and t^*, is represented as

$$\frac{\partial}{\partial z^*} p^*(z^*, t^*) = -q^*(z^*) f(t^*) \quad (6)$$

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with $q^*(z^*) = -\frac{\partial}{\partial z^*} p^*(z^*, 0), f(t^*) = 1 + A\sin(\omega t^*).$ (7)

In terms of these non-dimensional variables, eq. (1), (2), (7) reads

$$\alpha^2 \frac{\partial u}{\partial t} = 4q(z)f(t) - 2\frac{1}{r}\frac{\partial(r\tau)}{\partial r}, \qquad 0 < r < R(z,t), \quad (8)$$

Where $f(t) = 1 + A'sint, \theta = \frac{2\tau_y}{q_0 a}, \alpha^2 = \frac{\alpha^2\omega}{\frac{\mu}{\rho}}, q(z) = \frac{q^*(z^*)}{q_0}$, and

$$|\tau|^{1/2} = \sqrt{\theta} + \left|\frac{1}{2}\frac{\partial u}{\partial r}\right|^{1/2}, \text{ if } |\tau| \geq |\theta|; \frac{\partial u}{\partial r} = 0, \text{ if } |\tau| \leq |\theta|. \quad (9)$$

u=0 at r=R, τ is finite at r=0. (10)

The volumetric flow rate is given by

$$Q(z,t) = 4\int_0^{R(z,t)} \tau u(z,r,t)dr.$$

Solution of the problem:

Considering the Womersley parameter to be very small, the velocity u, shear stress τ as well as R_0 and R_p can be expressed in the following form

$u(z, r, t) = u_0(z, r, t) + \alpha^2 u_1(z, r, t) + \ldots$ (11)

$\tau(z, r, t) = \tau_0(z, r, t) + \alpha^2 \tau_1(z, r, t) + \ldots$ (12)

$R_p(z, t) = R_{0p}(z, t) + \alpha^2 R_{1p}(z, t) + \ldots$ (13)

$u_p(z, t) = u_{0p}(z, t) + \alpha^2 u_{1p}(z, t) + \ldots$ (14)

Using (11) and (12) in (8). we have

$$\frac{\partial}{\partial r}(r\tau_0) = -2rq(z)f(t) \quad (15)$$

$$\frac{\partial u_0}{\partial t} = \frac{2}{r}\frac{\partial}{\partial r}(r\tau_1) \quad (16)$$

Integrating (15) and using the boundary condition (10), we have

$$\tau_0 = -q(z)f(t)r, \quad (17)$$

Substituting Eqs. (11) and (12) into Eq. (9), we get

$$-\frac{\partial u_0}{\partial r} = 2\left[\theta + |\tau_0| - 2\sqrt{\theta|\tau_0|}\right], \quad (18)$$

$$-\frac{\partial u_1}{\partial r} = 2\tau_1\left[1 - \left(\frac{\theta}{|\tau_0|}\right)^{1/2}\right] \quad (19)$$

Substituting Eq. (17) into Eq. (18) and integrating using the boundary condition of Eq. (10), we obtain

$$u_0 = q(z)f(t)R^2\left[1 - \left(\frac{r}{R}\right)^2 - \left(\frac{8\Delta}{3\sqrt{R}}\right)\left[1 - \left(\frac{r}{R}\right)^{\frac{3}{2}}\right] + \left(2\Delta^2/R\right)[1 - (r/R)]\right], \quad (20)$$

Where $\Delta^2 = \theta/[q(z)f(t)]$. The plug core velocity can be obtain from Eq. (20) as

$$u_{0p} = q(z)f(t)R^2\left[1 - \left(\frac{R_{0p}}{R}\right)^2 - \left(\frac{8\Delta}{3\sqrt{R}}\right)\left[1 - \left(\frac{R_{0p}}{R}\right)^{\frac{3}{2}}\right] + \left(2\Delta^2/R\right)\left[1 - \left(R_{0p}/R\right)\right]\right], \quad (21)$$

Here R_{0p} is the first approximation plug core radius. Neglecting the term with α^2 and higher powers of α in Eq. (13), the expression for R_{0p} can be obtain from Eq. (17) as

$$R_{0p} = \frac{\theta}{|-q(z)|f(t)} = \Delta^2. \quad (22)$$

Similarly, the solution for τ_1, u_1, and u_{1p} can be obtained as

$$\tau_1 = q(z)f(t)\nabla R^3\left[2\left(r/R\right) - \left(r/R\right)^3 - \left(16\Delta/21\sqrt{R}\right)\left[7\left(\frac{r}{R}\right) - 4(r/R)^{5/2}\right] + (4\Delta^2/3R)\left[3\left(r/R\right) - 2\left(r/R\right)^2\right]\right], \quad (23)$$

$$u_1 = [q(z)f(t)\nabla R^4/16]\left[3 - 4\left(r/R\right)^2 + \left(r/R\right)^4 + \left(\frac{4\Delta}{\sqrt{R}}\right)\left(-\left(\frac{209}{74}\right) + \frac{4}{3}\left(r/R\right)^{\frac{3}{2}} + \frac{8}{3\left(r/R\right)^2} - \frac{170}{147\left(r/R\right)^{\frac{7}{2}}}\right) + \left(\frac{4\Delta^2}{R}\right)\left(\frac{13404}{315} - \frac{32}{9\left(r/R\right)^{\frac{3}{2}}} + 2\left(r/R\right)^2 + \frac{8}{63\left(r/R\right)^3}\right) + 4\left(\frac{\Delta}{\sqrt{R}}\right)^3\left(-\left(\frac{26}{5}\right) + 8\left(r/R\right)^{\frac{3}{2}} - 16/5\left(r/R\right)^{5/2}\right)\right], \quad (24)$$

$$u_1 = [u_1] \text{ at } r = R_{0p}, \quad (25)$$

where $\nabla = \left[\frac{1}{f(t)}\right][df(t)/dt]$, the shear stress on the wall τ_w is given by

$$\tau_w = q(z)f(t)R[1 + (\alpha^2\nabla R^2/8)[1 - \frac{16}{7}\left(\frac{\Delta}{\sqrt{R}}\right) + 4/3(\Delta^2/R)]], \quad (26)$$

The volumetric flow rate is given by

$$Q(t) = q(z)f(t)R^4[1 - \left(\frac{16}{7}\right)\left(\frac{\Delta}{\sqrt{R}}\right) + \left(\frac{4}{3}\right)\left(\frac{\Delta^2}{R}\right) + (\alpha^2\nabla R^2/4)[\left(\frac{2}{3}\right) - \left(\frac{426115}{161638}\right)\left(\frac{\Delta}{\sqrt{R}}\right) + \left(\frac{634}{105}\right)\left(\frac{\Delta^2}{R}\right) - (5168/315)\left(\frac{\Delta}{\sqrt{R}}\right)^3]]. \quad (27)$$

It may be noted that if we write $u = u_0 + \alpha^2 u_1$. From Eq. (27) for small $\Delta/\sqrt{R}$, we have

$$q(z) = \frac{Q_s}{R^4} + \frac{16}{7}\left(\frac{\theta Q_s}{R^5}\right)^{\frac{1}{2}} + \frac{64\theta}{49R}, \text{ where } R = R(z, t). \quad (28)$$

while computing q(z), one may take $Q_s = 1.0$. After q(z) is determined, Q(z, t) can be calculated from Eq.(27).

Results and discussion:

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The volumetric flow rate and the wall shear stress are the two important characteristics in the study of fluid flow through a stenosed artery. Using appropriate boundary conditions, analytical expressions for the velocity profile, volumetric flow rate and shear stress have been obtained. The expressions for volumetric flow rate and wall shear stress, given by (26) and (27) respectively have been numerically evaluated using MATLAB software for different values of relevant parameters.

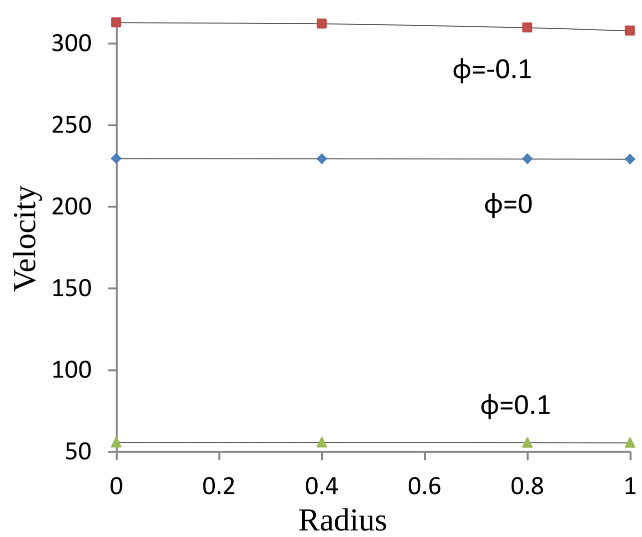

Fig. 2 Variation of velocity with radius for different values of ϕ

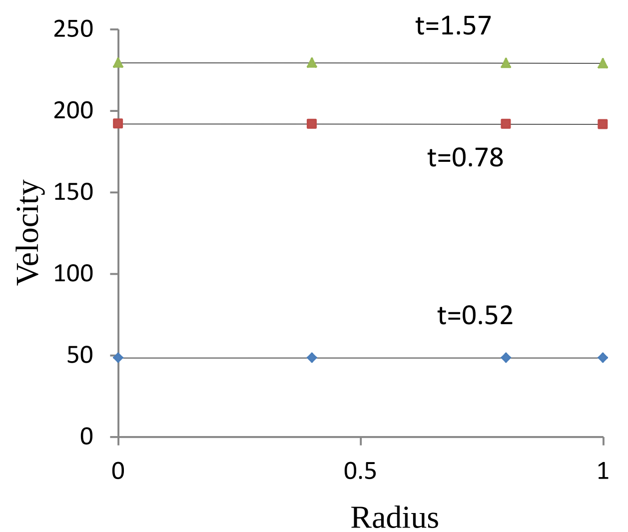

Fig.3 Variation of velocity with radius for different values of t

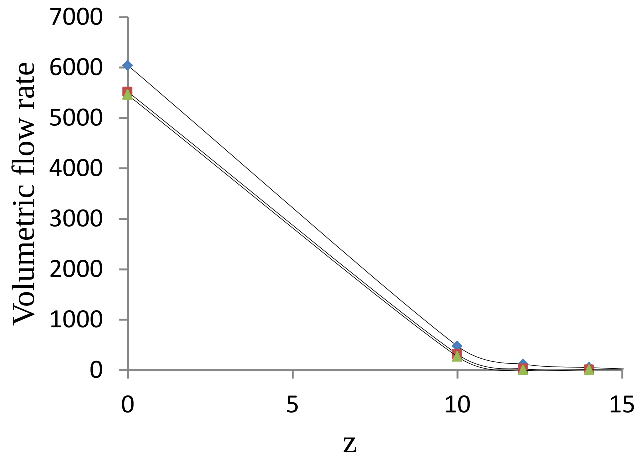

Fig. 4 Variation of volumetric flow rate with z for different values of ϕ

For the purpose of numerical computation of the quantities of interest, we have performed a thorough quantitative analysis, by taking the following values of the different parameters involved in the present study:
$a = 0.5mm$, $L = 30$, $L_0 = 10$, $d = 10$, $\theta = 0.05$, $A = 0.7$, $\delta = 0.1$, $\alpha^2 = 0.049$, $m = 2.0$, $T = 1.0$.

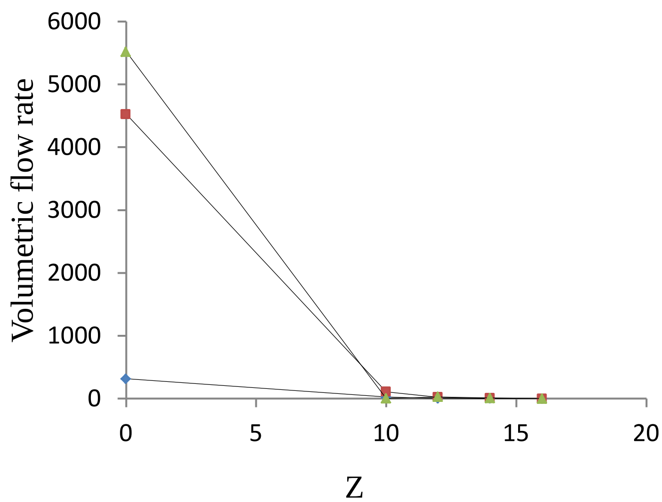

Fig. 5 Variation of volumetric flow rate with z for different values of t

Fig. (2-3) depicts the graph between the velocity profile and the radius of the artery. Fig. 2 shows that the velocity decreases when radius increases and also decreases when tapered angle and time is increases. Fig.3, shows that velocity decreases when radius increases and also decreases for the different values of time. Fig. (4,5) depicts the effect of volumetric flow rate with z. Fig. shows that the volumetric flow rate decreases when the axial distance z increases. Fig. (6-7) depicts that the effect of wall shear stress with axial distance z for different values of tapered angle and time. This shows that the wall shear stress increases in the starting when z varies from14 to 16 then decreases from 16 to 18 and decreases for decreasing value of time and also for decreasing values of tapered angle.

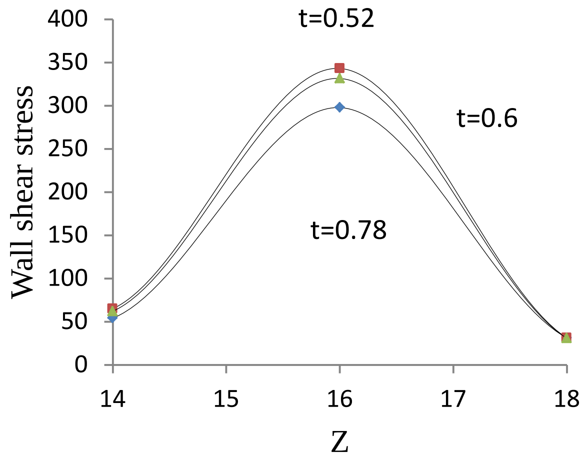

Fig. 6 Variation of wall shear stress with z for diffrernt values of t

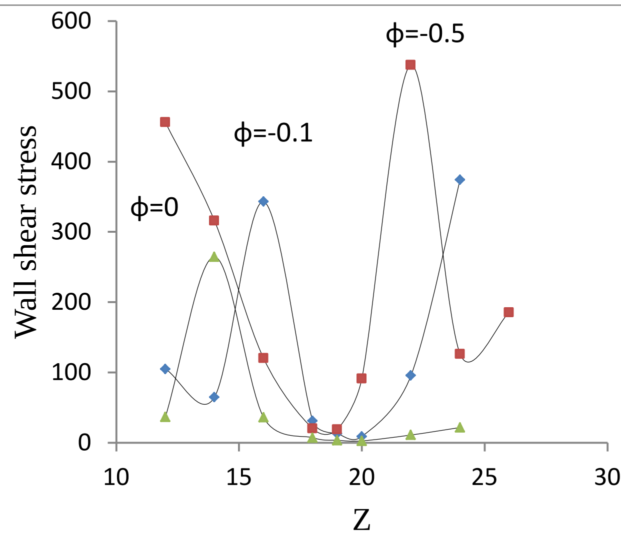

Fig.7. Variation if wall shear stress with Z for different values of ϕ

Conclusion:

From the above discussion, it is clear that the ratio of maximum height of stenosis and radius of normal artery and shear stress of the non-Newtonian fluid are strong parameter in fencing the blood flow. In this study it is obtained that the blood velocity decreases with the radial distribution for any given value of φ. It is also found that the velocity distribution of the two-fluid non- Newtonian Casson model are considerably higher than those of the Newtonian fluid models. It is also observed that the wall shear stress and the resistance to flow are significantly very low for the non- Newtonian casson model than those of the Newtonian fluid model. Hence, the non- Newtonian casson would be more useful than the Newtonian model to analyze the blood flow through stenosed tapered arteries. Hence from all the above discussions we can conclude that a careful choice of the fluid model will affect the flow characteristics and can be utilized for medical and engineering applications.

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