IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 3 Issue 3, March 2016.
www.ijiset.com

ISSN 2348 – 7968

Lyapunov-type inequality of a second-order system of dynamic equations on time scales

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Abstract In this note, we establish a Lyapunov-type inequality for the following second-order system of dynamic equations

$$(A(t)x^{\triangle}(t))^{\triangle} + B(t)x^{\sigma}(t) = 0$$

on the time scale interval $[a, b]_{\mathbf{T}} \equiv [a, b] \cap \mathbf{T}$ for some $a, b \in \mathbf{T}$, where $A(t), B(t)$ are real $n \times n$ symmetric matrix-valued functions on $[a, b]_{\mathbf{T}}$ and x is a real vector-valued function on $[a, b]_{\mathbf{T}}$.

AMS Subject Classification: 34K11, 34N05, 39A10.

Keywords: Lyapunov-type inequality; dynamic equation; time scale.

1 Introduction

In 1990, Hilger introduced in [7] the theory of time scales with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$, which has the topology that it inherits from the standard topology on $\mathbf{R}$. The two most popular examples are $\mathbf{R}$ and the integers $\mathbf{Z}$. For the time scale calculus and some related basic concepts, we refer the readers to the books by Bohner and Peterson [2,3] for further details.

In this note, we study Lyapunov-type inequality for the following second-order system of dynamic equations

$$(A(t)x^{\triangle}(t))^{\triangle} + B(t)x^{\sigma}(t) = 0 \tag{1.1}$$

Project Supported by NNSF of China (11461002 , 11461003) and NSF of Guangxi (2012GXNSFDA276040).
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on the time scale interval $[a,b]_{\mathbf{T}} \equiv [a,b] \cap \mathbf{T}$ for some $a, b \in \mathbf{T}$, where $A(t), B(t)$ are real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbf{T}}$ and x is a real vector-valued function on $[a,b]_{\mathbf{T}}$.

When $n = 1$, $A(t) = 1$ and $\mathbf{T} = \mathbf{R}$, (1.1) reduces to

$$x''(t) + B(t)x(t) = 0. \tag{1.2}$$

In 1907, Lyapunov [10] showed that if $B \in C([a,b], \mathbf{R})$ and $x(t) \not\equiv 0$ $(t \in [a,b])$ is a solution of (1.2) with $x(a) = x(b) = 0$, then the following classical Lyapunov inequality holds:

$$\int_a^b |B(t)|dt > \frac{4}{b-a}.$$

Moreover, the above inequality is optimal.

When $n = 1$, $A(t) = 1$ and $\mathbf{T} = \mathbf{Z}$, (1.1) reduces to

$$\Delta^2 x(m) + B(m)x(m+1) = 0. \tag{1.3}$$

In 1983, Cheng [5] investigated (1.3) under the assumptions $x(a) = x(b) = 0$ and $x(m) \not\equiv 0$ for $m \in \{a, a+1, \cdots, b\}$ and obtained the following Lyapunov inequality

$$\sum_{n=a}^{b-2} B(m) \geq \begin{cases} \frac{4(b-a)}{(b-a)^2-1}, & \text{if} \quad b-a-1 \quad \text{is even}, \\ \frac{4}{b-a}, & \text{if} \quad b-a-1 \quad \text{is odd}. \end{cases}$$

When $n = 1$ and $A(t) = 1$, (1.1) reduces to

$$x^{\Delta^2}(t) + B(t)x^{\sigma}(t) = 0. \tag{1.4}$$

In 2002, Bohner et al. [1] investigated (1.4) under the assumptions $x(a) = x(b) = 0$ and $x(t) \not\equiv 0$ for $t \in [a,b]_{\mathbf{T}}$ and $B : [a,b]_{\mathbf{T}} \longrightarrow (0, \infty)$ is rd-continuous, and obtained the following Lyapunov inequality

$$\int_a^b B(t)\Delta t \geq \frac{b-a}{C},$$

where $C = \max\{(t-a)(b-t) : t \in [a,b]_{\mathbf{T}}\}$.

In 2006, Wong et al. [12] investigated the following dynamic equation

$$(A(t)x^{\Delta}(t))^{\Delta} + B(t)x^{\sigma}(t) = 0 \tag{1.5}$$

under the assumptions $x(a) = x(b) = 0$ and $A \in C_{rd}([a,b]_{\mathbf{T}}, (0,\infty))$ is monotone and $B \in C_{rd}([a,b]_{\mathbf{T}}, \mathbf{R})$, and showed that if $x(t) \not\equiv 0$ for $t \in [a,b]_{\mathbf{T}}$ is a solution of (1.5), then

$$\int_a^b \max\{B(t), 0\}\Delta t \geq \begin{cases} \frac{A(a)(b-a)}{A(b)C}, & \text{if} \quad A \quad \text{is increasing}, \\ \frac{A(b)(b-a)}{A(a)C}, & \text{if} \quad A \quad \text{is decreasing}, \end{cases}$$

ISSN 2348 – 7968

where $C = \max\{(t-a)(b-t) : t \in [a,b]_{\mathbf{T}}\}$.

For some other related results on Lyapunov inequality, see, for example, [4,6,8, 11].

2 Main Result and its Proof

For any $x \in \mathbf{R}^n$ and any $A \in \mathbf{R}^{n\times n}$ (the space of real $n \times n$ matrices), denote by

$$|x| = \sqrt{x^T x} \quad \text{and} \quad |A| = \sup_{x\neq 0} \frac{|Ax|}{|x|}$$

the Euclidean norm of x and the matrix norm of A respectively, where C^T is transpose of a $n \times m$ matrix C. It is easy to show

$$|Ax| \le |A||x|$$

for any $x \in \mathbf{R}^n$ and any $A \in \mathbf{R}^{n\times n}$. Let $A \in \mathbf{R}^{n\times n}$ is a symmetric matrix, we say that A is semi-positive definite (resp. positive definite), written as $A \ge 0$ (resp. $A > 0$), if $x^T Ax \ge 0$ (resp. $x^T Ax > 0$) for all $x \in \mathbf{R}^n$. If A is semi-positive definite (resp. positive definite), then there exists a unique semi-positive definite matrix (resp. positive definite matrix), written as $\sqrt{A}$, such that $[\sqrt{A}]^2 = A$.

Lemma 1 Let $A(t) > 0$ be an integrable real $n \times n$ matrix-valued function, $u, v \in [a,b]_{\mathbf{T}}$ and x be an integrable real vector-valued function. Then

$$\int_v^u |\sqrt{A(t)}x^{\triangle}(t)|^2 \Delta t \ge \frac{|x(u)-x(v)|^2}{\int_v^u |[\sqrt{A(t)}]^{-1}|^2 \triangle t},$$

where C^{-1} denote the inverse matrix of C.

Proof Write $W = \frac{x(u)-x(v)}{\int_v^u |[\sqrt{A(t)}]^{-1}|^2 \triangle t}$. Then

$$\int_v^u |\sqrt{A(t)}x^{\triangle}(t) - [\sqrt{A(t)}]^{-1}W|^2 \triangle t \ge 0,$$

which implies

$$\begin{aligned}
&\int_v^u (\sqrt{A(t)}x^{\triangle}(t) - [\sqrt{A(t)}]^{-1}W)^T(\sqrt{A(t)}x^{\triangle}(t) - [\sqrt{A(t)}]^{-1}W)\triangle t \\
= &\int_v^u |\sqrt{A(t)}x^{\triangle}(t)|^2 \triangle t - \int_v^u W^T\{[\sqrt{A(t)}]^T\}^{-1}\sqrt{A(t)}x^{\triangle}(t)\triangle t \\
&- \int_v^u [x^{\triangle}(t)]^T[\sqrt{A(t)}]^T[\sqrt{A(t)}]^{-1}W\triangle t + \int_v^u |[\sqrt{A(t)}]^{-1}W|^2 \triangle t \\
= &\int_v^u |\sqrt{A(t)}x^{\triangle}(t)|^2 \triangle t - 2\int_v^u W^T x^{\triangle}(t)\triangle t + \int_v^u |[\sqrt{A(t)}]^{-1}W|^2 \triangle t \\
\ge &\ 0.
\end{aligned}$$

ISSN 2348 – 7968

That is

$$\begin{aligned}\int_v^u |\sqrt{A(t)}x^{\triangle}(t)|^2\triangle t &\geq 2\int_v^u W^T x^{\triangle}(t)\triangle t - \int_v^u |[\sqrt{A(t)}]^{-1}W|^2\triangle t\\ &\geq 2\int_v^u W^T x^{\triangle}(t)\triangle t - \int_v^u |[\sqrt{A(t)}]^{-1}|^2|W|^2\triangle t\\ &= 2W^T(x(u)-x(v)) - W^T W\int_v^u |[\sqrt{A(t)}]^{-1}|^2\triangle t\\ &= \frac{|x(u)-x(v)|^2}{\int_v^u |[\sqrt{A(t)}]^{-1}|^2\triangle t}.\end{aligned}$$

This completes the proof of Lemma 1.

We now can state and prove our main result.

Theorem 1 If (1.1) has a solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \not\equiv 0$ for $t \in [a,b]_{\mathbf{T}}$, then

$$\int_a^b |\sqrt{B(t)}|^2\Delta t \geq \frac{\int_a^b |[\sqrt{A(t)}]^{-1}|^2\triangle t}{f(d)},$$

where $f(d) = \max\{\int_a^t |[\sqrt{A(\tau)}]^{-1}|^2\triangle\tau \int_t^b |[\sqrt{A(\tau)}]^{-1}|^2\triangle\tau : t \in [a,b]_{\mathbf{T}}\}$.

Proof Since $|x(t)|^2$ is continuous in $[a,b]_{\mathbf{T}}$, we see that there exists an $c \in (a,b)_{\mathbf{T}}$ such that

$$|x(c)|^2 = \max\{|x(t)|^2 : t \in [a,b]_{\mathbf{T}}\} > 0.$$

Then we have

$$\begin{aligned}M\int_a^b |\sqrt{B(t)}|^2\Delta t &= \int_a^b |\sqrt{B(t)}|^2 M\Delta t \geq \int_a^b |\sqrt{B(t)}|^2|x^{\sigma}(t)|^2\Delta t\\ &\geq \int_a^b |\sqrt{B(t)}x^{\sigma}(t)|^2\Delta t = \int_a^b (\sqrt{B(t)}x^{\sigma}(t))^T\sqrt{B(t)}x^{\sigma}(t)\Delta t\\ &= \int_a^b (x^{\sigma}(t))^T[\sqrt{B(t)}]^T\sqrt{B(t)}x^{\sigma}(t)\Delta t = \int_a^b (x^{\sigma}(t))^T B(t)x^{\sigma}(t)\Delta t\\ &= -\int_a^b (x^{\sigma}(t))^T(A(t)x^{\triangle}(t))^{\triangle}\Delta t\\ &= -\int_a^b ((x(t))^T A(t)x^{\triangle}(t))^{\triangle}\Delta t + \int_a^b ((x(t))^T)^{\triangle}A(t)x^{\triangle}(t)\Delta t\\ &= \int_a^b ((x(t))^T)^{\triangle}A(t)x^{\triangle}(t)\Delta t = \int_a^b |\sqrt{A(t)}x^{\triangle}(t)|^2\Delta t\\ &= \int_a^c |\sqrt{A(t)}x^{\triangle}(t)|^2\Delta t + \int_c^b |\sqrt{A(t)}x^{\triangle}(t)|^2\Delta t\\ &\geq \frac{|x(c)-x(a)|^2}{\int_a^c |[\sqrt{A(t)}]^{-1}|^2\triangle t} + \frac{|x(b)-x(c)|^2}{\int_c^b |[\sqrt{A(t)}]^{-1}|^2\triangle t} \quad \text{(by Lemma 1)}\\ &= M\frac{\int_a^b |[\sqrt{A(t)}]^{-1}|^2\triangle t}{\int_a^c |[\sqrt{A(t)}]^{-1}|^2\triangle t \int_c^b |[\sqrt{A(t)}]^{-1}|^2\triangle t}\end{aligned}$$

$$\geq \quad M\frac{\int_a^b |[\sqrt{A(t)}]^{-1}|^2 \triangle t}{f(d)}.$$

That is

$$\int_a^b |\sqrt{B(t)}|^2 \Delta t \geq \frac{\int_a^b |[\sqrt{A(t)}]^{-1}|^2 \triangle t}{f(d)}.$$

This completes the proof of Theorem 1.

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