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Regular Elements of the Semigroup $B_X(D)$ defined by Semilattices of the Class $\Sigma_3(X,8)$ when $Z_7=\varnothing$

Giuli Tavdgiridze , Yasha Diasamidze

In this paper we give a full description of regular elements of the semigroup $B_X(D)$, which are defined by semilattices of the class $\Sigma_3(X,8)$. For the case where X-is a finite set we derive formulas by means of which we can calculate the numbers of regular elements of the respective semigroups. In this subsection it is assume that $Z_7=\varnothing$
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Definition 1.1. An element α taken from the semigroup $B_X(D)$ is colled a regular element of $B_X(D)$, if in $B_X(D)$ there exists an element β such that $\alpha\circ\beta\circ\alpha=\alpha$.

Definition 1.2. A one-to-one mapping φ between the complete $X-$ semilattices of unions D' and D'' is called a complete isomorphism if the condition

$$\varphi(D_1)=\bigcup_{T'\in D_1}\varphi(T')$$

is fulfilled for each nonempty subset D_1 of the semilattice D' (see [1],[2] Definition 6.3.2).

Definition 1.3. Let α be some binary relation of the semigroup $B_X(D)$. We say that a complete isomorphism φ between the complete semilatices of unions Q and D' is a complete $\alpha-$isomorphism if

a) $Q=V(D,\alpha)$;

b) $\varphi(\varnothing)=\varnothing$ for $\varnothing\in V(D,\alpha)$ and $\varphi(T)\alpha=T$ for all $T\in V(D,\alpha)$ (see [1],[2] Definition 6.3.3).

Theorem 1.1. Let D be a finite $X-$semilattice of unions and $\alpha\in B_X(D)$; $D(\alpha)$ be the set of those elements T of the semilattice $Q=V(D,\alpha)\setminus\{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. Then a binary relation α having a quasinormal representation of the form $\alpha=\bigcup_{T\in V(D,\alpha)}(Y_T^\alpha\times T)$ is a regular element of the semigroup $B_X(D)$ iff $V(D,\alpha)$ is a $XI-$semilattice of unions and for $\alpha-$isomorphism φ of the semilattice

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$V(D,\alpha)$ on some $X-$subsemilattice D' of the semilattice D the following conditions are satisfied:

a) $\bigcup_{T'\in \ddot{D}(\alpha)_T} Y_{T'}^{\alpha} \supseteq \varphi(T)$ for all $T \in D(\alpha)$;

b) $Y_T^{\alpha} \cap \varphi(T) \neq \varnothing$ for all nonlimiting element T of the set $\ddot{D}(\alpha)_T$ (see [1],[2] Theorem 6.3.3).

Theorem 1.2. Let R be the set of all regular elements of the semigroup $B_X(D)$. Then the following statements are true:

a) $R(D') \cap R(D'') = \varnothing$ for any $D', D'' \in \Sigma_{XI}(D)$ and $D' \neq D''$;

b) $R = \bigcup_{D'\in \Sigma_{XI}(D)} R(D')$;

c) if X is a finite set, then $|R| = \sum_{D'\in \Sigma_{XI}(D)} |R(D')|$ (see [1],[2] Theorem 6.3.6).

Theorem 1.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete $\alpha-$isomorphism φ of the semilattice $V(D,\alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the following conditions:

1) $\alpha = \varnothing$;
2) $\alpha = (Y_7^{\alpha} \times \varnothing) \cup (Y_{T'}^{\alpha} \times T')$, where $\varnothing \neq T' \in D$, $Y_{T'}^{\alpha} \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^{\alpha} \supseteq \varnothing$, $Y_{T'}^{\alpha} \cap \varphi(T') \neq \varnothing$;
3) $\alpha = (Y_7^{\alpha} \times \varnothing) \cup (Y_{T'}^{\alpha} \times T') \cup (Y_{T''}^{\alpha} \times T'')$, where $\varnothing \neq T' \subset T'' \in \breve{D}$, $Y_{T'}^{\alpha}, Y_{T''}^{\alpha} \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^{\alpha} \supseteq \varnothing$, $Y_7^{\alpha} \cup Y_{T'}^{\alpha} \supseteq \varphi(T')$, $Y_{T'}^{\alpha} \cap \varphi(T') \neq \varnothing$, $Y_{T''}^{\alpha} \cap \varphi(T'') \neq \varnothing$;
4) $\alpha = (Y_7^{\alpha} \times \varnothing) \cup (Y_{T'}^{\alpha} \times T') \cup (Y_{T''}^{\alpha} \times T'') \cup (Y_{T'''}^{\alpha} \times T''')$, where $\varnothing \neq T' \subset T'' \subset T''' \in D$, $Y_{T'}^{\alpha}, Y_{T''}^{\alpha}, Y_{T'''}^{\alpha} \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^{\alpha} \supseteq \varnothing$, $Y_7^{\alpha} \cup Y_{T'}^{\alpha} \supseteq \varphi(T')$, $Y_7^{\alpha} \cup Y_{T'}^{\alpha} \cup Y_{T''}^{\alpha} \supseteq \varphi(T'')$, $Y_{T'}^{\alpha} \cap \varphi(T') \neq \varnothing$, $Y_{T''}^{\alpha} \cap \varphi(T'') \neq \varnothing$, $Y_{T'''}^{\alpha} \cap \varphi(T''') \neq \varnothing$;
5) $\alpha = (Y_7^{\alpha} \times \varnothing) \cup (Y_T^{\alpha} \times T) \cup (Y_{T'}^{\alpha} \times T') \cup (Y_{T''}^{\alpha} \times T'') \cup (Y_0^{\alpha} \times \breve{D})$, where $Z_7 \neq T \subset T' \subset T'' \subset \breve{D}$, $Y_T^{\alpha}, Y_{T'}^{\alpha}, Y_{T''}^{\alpha}, Y_0^{\alpha} \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^{\alpha} \supseteq \varnothing$, $Y_7^{\alpha} \cup Y_T^{\alpha} \supseteq \varphi(T)$, $Y_7^{\alpha} \cup Y_T^{\alpha} \cup Y_{T'}^{\alpha} \supseteq \varphi(T')$, $Y_7^{\alpha} \cup Y_T^{\alpha} \cup Y_{T'}^{\alpha} \cup Y_{T''}^{\alpha} \supseteq \varphi(T'')$, $Y_T^{\alpha} \cap \varphi(T) \neq \varnothing$, $Y_{T'}^{\alpha} \cap \varphi(T') \neq \varnothing$, $Y_{T''}^{\alpha} \cap \varphi(T'') \neq \varnothing$, $Y_0^{\alpha} \cap \breve{D} \neq \varnothing$;
6) $\alpha = (Y_7^{\alpha} \times \varnothing) \cup (Y_{T'}^{\alpha} \times T') \cup (Y_{T''}^{\alpha} \times T'') \cup (Y_{T' \cup T''}^{\alpha} \times (T' \cup T''))$, where $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_{T'}^{\alpha}, Y_{T''}^{\alpha} \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^{\alpha} \cup Y_{T'}^{\alpha} \supseteq \varphi(T')$, $Y_7^{\alpha} \cup Y_{T''}^{\alpha} \supseteq \varphi(T'')$, $Y_{T'}^{\alpha} \cap \varphi(T') \neq \varnothing$, $Y_{T''}^{\alpha} \cap \varphi(T'') \neq \varnothing$;

7) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)\cup\left(Y_{T''\cup T'''}^\alpha\times\left(T''\cup T'''\right)\right)$, where, $\varnothing\neq T'\subset T''$, $\varnothing\neq T'\subset T'''$, $T''\setminus T'''\neq\varnothing$, $T'''\setminus T''\neq\varnothing$ $Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$ $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$ $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$.

8) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_6,Z_5\}$, $Y_T^\alpha,Y_4^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$, $Y_7^\alpha\cup Y_T^\alpha\supseteq\varphi(T)$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\supseteq\varphi(Z_4)$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq\varphi(Z_2)$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq\varphi(Z_1)$, $Y_T^\alpha\cap\varphi(T)\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$ $Y_2^\alpha\cap Z_2\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$

9) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$ $Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$ $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

10) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)$, where, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T'\cup T''\subset T'''$ $Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_7^\alpha\cup Y_{T''}^\alpha\supseteq\varphi(T'')$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_{T''}^\alpha\cap\varphi(T'')\neq\varnothing$, $Y_{T'''}^\alpha\cap\varphi(T''')\neq\varnothing$;

11) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_2,Z_1\}$, $Y_6^\alpha,Y_5^\alpha,Y_T^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_T^\alpha\supseteq\varphi(T)$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_5^\alpha\cap Z_5\neq\varnothing$ $Y_T^\alpha\cap T\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

12) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)\cup\left(Y_{T'\cup T''\cup T'''}^\alpha\times\left(T'\cup T''\cup T'''\right)\right)$, where $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T''\subset T'''$, $\left(T'\cup T''\right)\setminus T'''\neq\varnothing$, $T'''\setminus\left(T'\cup T''\right)\neq\varnothing$, , $Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha,Y_{T'\cup T''\cup T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_7^\alpha\cup Y_{T''}^\alpha\supseteq\varphi(T'')$ $Y_7^\alpha\cup Y_{T''}^\alpha\cup Y_{T'''}^\alpha\supseteq\varphi(T''')$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_{T''}^\alpha\cap\varphi(T'')\neq\varnothing$ $Y_{T'''}^\alpha\cap\varphi(T''')\neq\varnothing$;

13) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Z_4\subset Z_2$, $Z_1\setminus Z_2\neq\varnothing$, $Z_2\setminus Z_1\neq\varnothing$, , $Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$ $Y_1^\alpha\cap Z_1\neq\varnothing$;

14) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where, $Z_6\subset Z_4$, $Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$ $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

15) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$ $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$ $Y_1^\alpha\cap Z_1\neq\varnothing$;

16) $\alpha=\left(Y_7^\alpha\times\varnothing\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z\ \right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where, $Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$ $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$;

(see Theorem 1.1 in[3])

Lemma 1.1

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a) $\left|R^*(Q_1)\right| = 1$;

b) $\left|R^*(Q_2)\right| = m_0 \cdot \left(2^{|T'|} - 1\right) \cdot 2^{|X \setminus T'|}$;

c) $\left|R^*(Q_3)\right| = m_0 \cdot \left(2^{|T'|} - 1\right) \cdot \left(3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}\right) \cdot 3^{|X \setminus T''|}$;

d) $\left|R^*(Q_4)\right| = m_0 \cdot \left(2^{|T'|} - 1\right) \cdot \left(3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}\right) \cdot \left(4^{|T''' \setminus T''|} - 3^{|T''' \setminus T''|}\right) \cdot 4^{|X \setminus T'''|}$;

e) $\left|R^*(Q_5)\right| = m_0 \cdot \left(2^{|T|} - 1\right) \cdot \left(3^{|T' \setminus T|} - 2^{|T' \setminus T|}\right) \cdot \left(4^{|T'' \setminus T'|} - 3^{|T'' \setminus T'|}\right) \cdot \left(5^{|\breve{D} \setminus T''|} - 4^{|\breve{D} \setminus T''|}\right) \cdot 5^{|X \setminus \breve{D}|}$;

f) $\left|R^*(Q_6)\right| = 2 \cdot m_0 \cdot \left(2^{|T' \setminus T''|} - 1\right) \cdot \left(2^{|T'' \setminus T'|} - 1\right) \cdot 4^{|X \setminus (T' \cup T'')|}$;

g) $\left|R^*(Q_7)\right| = 2 \cdot m_0 \cdot \left(2^{|T'|} - 1\right) \cdot 2^{|(T'' \cap T''') \setminus T'|} \cdot \left(3^{|T'' \setminus T'''|} - 2^{|T'' \setminus T'''|}\right) \cdot \left(3^{|T''' \setminus T''|} - 2^{|T''' \setminus T''|}\right) \cdot 5^{|X \setminus (T'' \cup T''')|}$;

h) $\left|R^*(Q_8)\right| = 2 \cdot m_0 \cdot \left(2^{|T|} - 1\right) \cdot \left(3^{|Z_4 \setminus T|} - 2^{|Z_4 \setminus T|}\right) \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot \left(4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}\right) \cdot \left(4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}\right) \cdot 6^{|X \setminus \breve{D}|}$;

i) $\left|R^*(Q_9)\right| = 2 \cdot m_0 \cdot \left(2^{|Z_5|} - 1\right) \cdot 2^{|(Z_3 \cap Z_4) \setminus Z_5|} \cdot \left(3^{|Z_3 \setminus Z_4|} - 2^{|Z_3 \setminus Z_4|}\right) \cdot \left(3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}\right) \cdot \left(6^{|\breve{D} \setminus (Z_3 \cup Z_4)|} - 5^{|\breve{D} \setminus (Z_3 \cup Z_4)|}\right) \cdot 6^{|X \setminus \breve{D}|}$;

j) $\left|R^*(Q_{10})\right| = 2 \cdot m_0 \cdot \left(2^{|T' \setminus T''|} - 1\right) \cdot \left(2^{|T'' \setminus T'|} - 1\right) \cdot \left(5^{|T''' \setminus (T' \cup T'')|} - 4^{|T''' \setminus (T' \cup T'')|}\right) \cdot 5^{|X \setminus T'''|}$;

k) $\left|R^*(Q_{11})\right| = 2 \cdot m_0 \cdot \left(2^{|T \setminus T'|} - 1\right) \cdot \left(2^{|T' \setminus T|} - 1\right) \cdot \left(5^{|T'' \setminus Z_4|} - 4^{|T'' \setminus Z_4|}\right) \cdot \left(6^{|\breve{D} \setminus T''|} - 5^{|\breve{D} \setminus T|''}\right) \cdot 6^{|X \setminus \breve{D}|}$;

l) $\left|R^*(Q_{12})\right| = m_0 \cdot \left(2^{|T' \setminus T'''|} - 1\right) \cdot \left(2^{|T'' \setminus T|} - 1\right) \cdot \left(3^{|T''' \setminus (T' \cup T'')|} - 2^{|T''' \setminus (T' \cup T'')|}\right) \cdot 6^{|X \setminus (T' \cup T'' \cup T''')|}$;

m) $\left|R^*(Q_{13})\right| = m_0 \cdot \left(2^{|Z_5|} - 1\right) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_5|} \cdot \left(3^{|Z_3 \setminus Z_4|} - 2^{|Z_3 \setminus Z_4|}\right) \cdot \left(3^{|Z_4 \setminus Z_2|} - 2^{|Z_4 \setminus Z_2|}\right) \cdot \left(4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|}$;

n) $\left|R^*(Q_{14})\right| = m_0 \cdot \left(2^{|Z_5 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}\right) \cdot \left(7^{|\breve{D} \setminus Z_1|} - 6^{|\breve{D} \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|}$;

o) $\left|R^*(Q_{15})\right| = 4 \cdot m_0 \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot 4^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot \left(5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}\right) \cdot 7^{|X \setminus \breve{D}|}$

p) $\left|R^*(Q_{16})\right| = \left(2^{|Z_6 \setminus Z_3|} - 1\right) \cdot 2^{|Z_5 \setminus Z_4|} \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_3 \setminus Z_2|} - 2^{|Z_3 \setminus Z_2|}\right) \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot 8^{|X \setminus \breve{D}|}$

(see Lemma 1.1 in [3])

1) **Lemma 1.2.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_3(X, 8)$ and $Z_7 = \varnothing$. Then $\left|R^*(Q_1)\right| = 1$.

(see Lemma 1.2 in [3])

2) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $b)$ of the Theorem 1.3 In this case we have $Q_2 = \{\varnothing, T'\}$, By definition of the semilattice D follows that

$$Q_2 \vartheta_{XI} = \left\{\{\varnothing, \breve{D}\}, \{\varnothing, Z_6\}, \{\varnothing, Z_5\}\ \{\varnothing, Z_4\}, \{\varnothing, Z_3\}, \{\varnothing, Z_2\}, \{\varnothing, Z_1\}\right\}$$

It is easy to see $\left|\Phi(Q_2, Q_2)\right| = 1$ and $\left|\Omega(Q_2)\right| = 7$. Assume that

$$D_1' = \{\varnothing, \breve{D}\},\ D_2' = \{\varnothing, Z_6\},\ D_3' = \{\varnothing, Z_5\},\ D_4' = \{\varnothing, Z_4\},\ D_5' = \{\varnothing, Z_3\},\ D_6' = \{\varnothing, Z_2\}, D_7' = \{\varnothing, Z_1\}$$

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Lemma 1.3. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

$$\left|R^*(Q_2)\right|=7\cdot\left(2^{|\breve{D}|}-1\right)\cdot 2^{|X\setminus\breve{D}|}.$$

(see Lemma 1.3 in [3])

$c)$ Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $c)$ of the Theorem 1.3 In this case we have $Q_2=\{\varnothing,T',T''\}$, where $T',T''\in D$ and $\varnothing\neq T'\subset T''$. By definition of the semilattice D follows that

$$\begin{aligned}Q_3\vartheta_{XI}=\{&\{\varnothing,Z_1,\breve{D}\},\{\varnothing,Z_2,\breve{D}\},\{\varnothing,Z_3,\breve{D}\},\{\varnothing,Z_4,D\},\{\varnothing,Z_5,\breve{D}\},\{\varnothing,Z_6,\breve{D}\},\{\varnothing,Z_6,Z_4\},\\&\{\varnothing,Z_6,Z_2\},\{\varnothing,Z_6,Z_1\},\{\varnothing,Z_5,Z_4\},\{\varnothing,Z_5,Z_3\},\{\varnothing,Z_5,Z_2\},\{\varnothing,Z_5,Z_1\},\{\varnothing,Z_4,Z_2\},\\&\{\varnothing,Z_4,Z_1\},\{\varnothing,Z_3,Z_1\}\}\end{aligned}$$

It is easy to see $|\Phi(Q_3,Q_3)|=1$ and $|\Omega(Q_3)|=16$ .assume that

$$\begin{aligned}&D_1'=\{\varnothing,Z_1,\breve{D}\},\ D_2'=\{\varnothing,Z_2,\breve{D}\},\ D_3'=\{\varnothing,Z_3,\breve{D}\},\ D_4'=\{\varnothing,Z_4,D\},\ D_5'=\{\varnothing,Z_5,\breve{D}\},\\&D_6'=\{\varnothing,Z_6,\breve{D}\},\ D_7'=\{\varnothing,Z_6,Z_4\},\ D_8'=\{\varnothing,Z_6,Z_2\},\ D_9'=\{\varnothing,Z_6,Z_1\},\ D_{10}'=\{\varnothing,Z_5,Z_4\},\\&D_{11}'=\{\varnothing,Z_5,Z_3\},\ D_{12}'=\{\varnothing,Z_5,Z_2\},\ D_{13}'=\{\varnothing,Z_5,Z_1\},\ D_{14}'=\{\varnothing,Z_4,Z_2\},\ D_{15}'=\{\varnothing,Z_4,Z_1\},\\&D_{16}'=\{\varnothing,Z_3,Z_1\}\end{aligned}$$

Lemma 1.4. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

$$\begin{aligned}\left|R^*(Q_3)\right|=&\sum_{i=1}^{6}\left|R(D_i')\right|+\left|R(D_1')\cap R(D_5')\right|-\\&-\left|R(D_1')\cap R(D_3')\right|-\left|R(D_1')\cap R(D_4')\right|-\left|R(D_2')\cap R(D_4')\right|-\\&-\left|R(D_3')\cap R(D_5')\right|-\left|R(D_4')\cap R(D_5')\right|-\left|R(D_4')\cap R(D_6')\right|\end{aligned}$$

(see Lemma 1.4 in [3])

Lemma 1.5. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

$$\begin{aligned}\left|R^*(Q_3)\right|=&16\cdot\left(2^{|Z_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&+16\cdot\left(2^{|Z_2|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&+16\cdot\left(2^{|Z_3|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&+16\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&+16\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&+16\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&+16\cdot 2^{|Z_1\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}+\\&-16\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_3|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}-\\&-16\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}-\\&-16\cdot 2^{|Z_2\setminus Z_4|}\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}-\\&-16\cdot 2^{|Z_3\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\breve{D}|}-\\&-16\cdot 2^{|Z_4\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}-\\&-16\cdot 2^{|Z_4\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}\end{aligned}$$

(see Lemma 1.5 in [3])

d′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition d) of the Theorem 1.3 In this case we have $Q_4=\{\varnothing,T',Z,Z'\}$, where $T',Z,Z'\in D$ and $\varnothing\neq T'\subset Z\subset Z'$. By definition of the semilattice D follows that

$$\begin{array}{l}Q_4\vartheta_{XI}=\{\{\varnothing,Z_6,Z_4,\breve{D}\},\ \varnothing\{Z_7,Z_6,Z_2,D\},\ \{\varnothing,Z_6,Z_1,\breve{D}\},\ \{\varnothing,Z_5,Z_4,\breve{D}\},\ \{\varnothing,Z_5,Z_3,\breve{D}\},\\ \{\varnothing,Z_5,Z_2,\breve{D}\},\ \{\varnothing,Z_5,Z_1,\breve{D}\},\ \{\varnothing,Z_4,Z_2,D\},\ \{\varnothing,Z_4,Z_1,\breve{D}\},\ \{\varnothing,Z_3,Z_1,\breve{D}\}\\ \{\varnothing,Z_5,Z_4,Z_2\},\ \{\varnothing,Z_5,Z_4,Z_1\},\ \{\varnothing,Z_5,Z_3,Z_1\},\ \{\varnothing,Z_6,Z_4,Z_2\},\ \{\varnothing,Z_6,Z_4,Z_1\},\end{array}$$

It is easy to see $|\Phi(Q_4,Q_4)|=1$ and $|\Omega(Q_4)|=15$. Assume that

$$\begin{array}{l}D_1'=\{\varnothing,Z_6,Z_4,\breve{D}\},D_2'=\{\varnothing,Z_6,Z_2,D\},D_3'=\{\varnothing,Z_6,Z_1,\breve{D}\},D_4'=\{\varnothing,Z_5,Z_4,\breve{D}\},\\ D_5'=\{\varnothing,Z_5,Z_3,\breve{D}\},D_6'=\{\varnothing,Z_5,Z_2,\breve{D}\},D_7'=\{\varnothing,Z_5,Z_1,\breve{D}\},D_8'=\{\varnothing,Z_4,Z_2,D\},\\ D_9'=\{\varnothing,Z_4,Z_1,\breve{D}\},D_{10}'=\{\varnothing,Z_3,Z_1,\breve{D}\},D_{11}'=\{\varnothing,Z_6,Z_4,Z_2\},D_{12}'=\{\varnothing,Z_6,Z_4,Z_1\}\\ D_{13}'=\{\varnothing,Z_5,Z_4,Z_2\},D_{14}'=\{\varnothing,Z_5,Z_4,Z_1\},D_{15}'=\{\varnothing,Z_5,Z_3,Z_1\},\end{array}.$$

Lemma 1.6. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

$$\begin{array}{l}|R^*(Q_4)|=\sum_{i=1}^{10}|R(D_i')|-|R(D_1')\cap R(D_2')|-|R(D_1')\cap R(D_3')|-\\ -|R(D_2')\cap R(D_8')|-|R(D_3')\cap R(D_9')|-|R(D_4')\cap R(D_6')|-\\ -|R(D_4')\cap R(D_7')|-|R(D_6')\cap R(D_8')|-|R(D_7')\cap R(D_9')|-\\ -|R(D_7')\cap R(D_{10}')|\end{array}$$

(see Lemma 1.6 in [3])

Lemma 1.7 Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

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$$
\begin{aligned}
\left|R^*(Q_4)\right| = {} & 15\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\\
& +15\cdot\left(2^{|Z_3\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
& -15\cdot 3^{|Z_6\setminus Z_6|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 3^{|Z_2\setminus Z_4|}\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 3^{|Z_6\setminus Z_6|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 3^{|Z_1\setminus Z_4|}\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 3^{|Z_4\setminus Z_6|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 3^{|Z_2\setminus Z_2|}\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_4\setminus Z_6|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 3^{|Z_1\setminus Z_1|}\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_5\setminus Z_5|}\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 3^{|Z_2\setminus Z_4|}\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_5\setminus Z_5|}\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 3^{|Z_1\setminus Z_4|}\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_5\setminus Z_5|}\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 3^{|Z_1\setminus Z_3|}\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_4\setminus Z_5|}\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 3^{|Z_2\setminus Z_2|}\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_4\setminus Z_5|}\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 3^{|Z_1\setminus Z_1|}\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}-\\
& -15\cdot 2^{|Z_3\setminus Z_5|}\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 3^{|Z_1\setminus Z_1|}\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}
\end{aligned}
$$

(see Lemma 1.7 in [3])

e′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $e)$ of the Theorem 1.3 In this case we have $Q_5=\{\varnothing,T,T',T'',\breve{D}\}$, where $T,T',T''\in D$ and $T\subset T'\subset T''\in D$. By definition of the semilattice D follows that

$$
\begin{aligned}
Q_5\vartheta_{XI}=\big\{&\{\varnothing,Z_6,Z_4,Z_2,\breve{D}\},\{\varnothing,Z_6,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_2,\breve{D}\},\\
&\{\varnothing,Z_5,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_3,Z_1,\breve{D}\}\big\}.
\end{aligned}
$$

It is easy to see $|\Phi(Q_5,Q_5)|=1$ and $|\Omega(Q_5)|=5$. Assume that

$$
\begin{aligned}
&D_1'=\{\varnothing,Z_6,Z_4,Z_2,\breve{D}\},\ D_2'=\{\varnothing,Z_6,Z_4,Z_1,\breve{D}\},\ D_3'=\{\varnothing,Z_5,Z_4,Z_2,\breve{D}\},\\
&D_4'=\{\varnothing,Z_5,Z_4,Z_1,\breve{D}\},\ D_5'=\{\varnothing,Z_5,Z_3,Z_1,\breve{D}\}
\end{aligned}
$$

Lemma 1.8. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

$$
\left|R^*(Q_5)\right|=\left|R(D_1')\right|+\left|R(D_2')\right|+\left|R(D_3')\right|+\left|R(D_4')\right|+\left|R(D_5')\right|
$$

(see Lemma 1.8 in [3])

Lemma 1.9. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

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$$\begin{aligned}\left|R^*(Q_5)\right| &= 5\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}+\\ &+5\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}+\\ &+5\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}+\\ &+5\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}+\\ &+5\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

(see Lemma 1.9 in [3])

f′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $f)$ of the Theorem 1.3. In this case we have $Q_6=\{T,T',T'',T'\cup T''\}$, where $T,T',T''\in D$ and $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$. By definition of the semilattice D follows that

$$Q_6\vartheta_{XI}=\left\{\{\varnothing,Z_2,Z_1,\breve{D}\},\ \{\varnothing,Z_6,Z_5,Z_4\},\{\varnothing,Z_6,Z_3,Z_1\},\ \{\varnothing,Z_4,Z_3,Z_1\},\ \varnothing\{Z_7,Z_3,Z_2,\breve{D}\}\right\}$$

It is easy to see $|\Phi(Q_6,Q_6)|=2$ and $|\Omega(Q_6)|=10$. Assume that

$$\begin{aligned}&D_1'=\{\varnothing,Z_2,Z_1,\breve{D}\},\ D_2'=\{\varnothing,Z_1,Z_2,\breve{D}\},\ D_3'=\{\varnothing,Z_6,Z_5,Z_4\},\ D_4'=\{\varnothing,Z_5,Z_6,Z_4\},\\ &D_5'=\{\varnothing,Z_6,Z_3,Z_1\},\ D_6'=\{\varnothing,Z_3,Z_6,Z_1\},\ D_7'=\{\varnothing,Z_4,Z_3,Z_1\},D_8'=\{\varnothing,Z_3,Z_4,Z_1\},\\ &D_9'=\{\varnothing,Z_3,Z_2,\breve{D}\},D_{10}'=\{\varnothing,Z_2,Z_3,\breve{D}\},\end{aligned}$$

Lemma 1.10. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If by $R^*(Q_6)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition $f)$ of the Theorem 1.3, then

$$\begin{aligned}\left|R^*(Q_6)\right|&=\sum_{i=1}^{10}\left|R(D_i')\right|-\left|R(D_1')\cap R(D_{10}')\right|-\left|R(D_2')\cap R(D_9')\right|-\\ &-\left|R(D_3')\cap R(D_5')\right|-\left|R(D_4')\cap R(D_6')\right|-\left|R(D_5')\cap R(D_7')\right|-\left|R(D_6')\cap R(D_8')\right|-\\ &-\left|R(D_7')\cap R(D_{10}')\right|-\left|R(D_8')\cap R(D_9')\right|\end{aligned}$$

(see Lemma 1.10 in [3])

Lemma 1.11. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then

$$\begin{aligned}\left|R^*(Q_6)\right|&=10\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+20\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}+\\ &+10\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_4|}+20\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 4^{|X\setminus Z_1|}+\\ &+10\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\ -5\cdot 2^{|Z_2\setminus\breve{D}|}&\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus Z_1|}-5\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}-\\ -5\cdot 2^{|Z_6\setminus Z_1|}&\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus Z_4|}\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}-5\cdot 2^{|Z_3\setminus Z_4|}\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 2^{|Z_6\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}-\\ -5\cdot 2^{|Z_4\setminus Z_1|}&\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 4^{|X\setminus Z_1|}-5\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 2^{|Z_4\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}-\\ -5\cdot 2^{|Z_2\setminus Z_1|}&\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}-5\cdot 2^{|Z_3\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\end{aligned}$$

(see Lemma 1.11 in [3])

g′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $g)$ of the Theorem 1.3 In this case we have $\{T,T',T'',T''',T''\cup T'''\}$, where $T,T',T'',T'''\in D$, $T\subset T'\subset T''$, $T\subset T'\subset T'''$, $T\setminus T'\neq\varnothing$ and $T'\setminus T\neq\varnothing$. By definition of the semilattice D follows that

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$$Q_7\vartheta_{XI}=\left\{\{\varnothing,Z_4,Z_2,Z_1,\breve{D}\},\{\varnothing,Z_6,Z_2,Z_1,\breve{D}\},\ \{\varnothing,Z_5,Z_2,Z_1,\breve{D}\},\right.$$
$$\left.\{\varnothing,Z_5,Z_3,Z_2,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_3,Z_1\}\right\}$$

It is easy to see $|\Phi(Q_7,Q_7)|=2$ and $|\Omega(Q_6)|=7$. assume

$$D_1'=\{\varnothing,Z_4,Z_2,Z_1,\breve{D}\},D_2'=\{\varnothing,Z_4,Z_1,Z_2,\breve{D}\},\ D_3'=\{\varnothing,Z_6,Z_2,Z_1,\breve{D}\},\ D_4'=\{\varnothing,Z_6,Z_1,Z_2,\breve{D}\},$$
$$D_5'=\{\varnothing,Z_5,Z_2,Z_1,\breve{D}\},D_6'=\{\varnothing,Z_5,Z_1,Z_2,\breve{D}\},D_7'=\{\varnothing,Z_5,Z_3,Z_2,\breve{D}\},D_8'=\{\varnothing,Z_5,Z_2,Z_3,\breve{D}\},$$
$$D_9'=\{\varnothing,Z_5,Z_4,Z_3,Z_1\},\ D_{10}'=\{\varnothing,Z_5,Z_3,Z_4,Z_1\}$$

Lemma 1.12. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If by $R^*(Q_7)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition $g)$ of the Theorem 1.3 then

$$\begin{aligned}|R^*(Q_7)|=\sum_{i=1}^{7}|R(D_i')|-|R(D_1')\cap R(D_3')|-|R(D_1')\cap R(D_5')|-\\-|R(D_2')\cap R(D_4')|-|R(D_2')\cap R(D_6')|-|R(D_2')\cap R(D_7')|-\\-|R(D_5')\cap R(D_8')|-|R(D_7')\cap R(D_{10}')|-|R(D_8')\cap R(D_9')|\end{aligned}$$

(see Lemma 1.12 in [3])

Lemma 1.13. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_3(X,8)$ *and* $Z_7\neq\varnothing$. *Then*

$$\begin{aligned}|R^*(Q_7)|=&10\cdot\left(2^{|Z_4|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}+\\
&+10\cdot\left(2^{|Z_6|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_6|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}+\\
&+10\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_5|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}+\\
&+10\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_3)\setminus Z_5|}\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}+\\
&+10\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_4\cap Z_3)\setminus Z_5|}\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot5^{|X\setminus\breve{D}|}+\\
&-5\cdot2^{|Z_4\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot3^{|Z_2\setminus\breve{D}|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot3^{|Z_1\setminus\breve{D}|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_4\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot3^{|Z_2\setminus\breve{D}|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot3^{|Z_1\setminus\breve{D}|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_4\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot3^{|Z_1\setminus\breve{D}|}\cdot\left(3^{\|Z_1\setminus Z_2\|}-2^{\|Z_1\setminus Z_2\|}\right)\cdot3^{|Z_2\setminus\breve{D}|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_4\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot3^{|Z_2\setminus\breve{D}|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot3^{|Z_1\setminus\breve{D}|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_4\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot3^{|Z_1\setminus\breve{D}|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot3^{|Z_2\setminus\breve{D}|}\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_5\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_1)\setminus Z_5|}\cdot3^{|Z_2\setminus\breve{D}|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot3^{|Z_1\setminus\breve{D}|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_5\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_3)\setminus Z_5|}\cdot3^{|Z_3\setminus\breve{D}|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot3^{|Z_2\setminus Z_1|}\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot5^{|X\setminus\breve{D}|}-\\
&-5\cdot2^{|Z_5\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot2^{|(Z_2\cap Z_3)\setminus Z_5|}\cdot3^{|Z_2\setminus Z_1|}\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot3^{|Z_3\setminus\breve{D}|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot5^{|X\setminus\breve{D}|}\end{aligned}$$

(see Lemma 1.13 in [3])

h′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $h)$ of the Theorem 1.3 In this case we have $Q_8=\{Z_7,T,Z_4,Z_2,Z_1,\breve{D}\}$, where $T\in\{Z_5,Z_6\}$. By definition of the semilattice D follows that

$$Q_8\vartheta_{XI}=\left\{\{\varnothing,Z_6,Z_4,Z_2,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_2,Z_1,\breve{D}\}\right\}.$$

It is easy to see $|\Phi(Q_8,Q_8)|=2$ and $|\Omega(Q_8)|=2$. assume

$$D_1' = \{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\},\ D_2' = \{\varnothing, Z_6, Z_4, Z_1, Z_2, \breve{D}\},$$
$$D_3' = \{\varnothing, Z_5, Z_4, Z_2, Z_1, \breve{D}\},\ D_4' = \{\varnothing, Z_5, Z_4, Z_1, Z_2, \breve{D}\}.$$

Lemma 1.14. Let X be a finite set, $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If by $R^*(Q_8)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition $h)$ of the Theorem 1.3, then

$$\begin{aligned}\left|R^*(Q_8)\right| &= \left|R(D_1')\right| + \left|R(D_2')\right| + \left|R(D_3')\right| + \left|R(D_4')\right| = \\ &= 4\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot \\ &\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|} + 4\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot \\ &\cdot 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}.\end{aligned}$$

(see Lemma 1.14 in [3])

i′) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition $p)$ of the Theorem 1.3). in this case we have $\{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, By definition of the semilattice D follows that $Q_9\vartheta_{XI} = \{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$ It is easy to see $\left|\Phi(Q_9, Q_9)\right| = 2$ and $\left|\Omega(Q_9)\right| = 1$. If $D_1' = \{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, then $R^*(Q_9) = R(D_1')$, $\left|R^*(Q_9)\right| = \left|R(D_1')\right|$ and

$$\left|R^*(Q_9)\right| = \left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot 6^{|X\setminus\breve{D}|}$$

j′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $j)$ of the Theorem 1.3 In this case we have $Q_{10} = \{T, T', T'', T'\cup T'', Z\}$, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T'\cup T''\subset Z$. By definition of the semilattice D follows that

$$\begin{aligned}Q_{10}\vartheta_{XI} = \{&\{\varnothing, Z_6, Z_5, Z_4, \breve{D}\}, \{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_1, \breve{D}\}, \\ &\{\varnothing, Z_6, Z_5, Z_4, Z_2\},\ \{\varnothing, Z_6, Z_5, Z_4, Z_1\}\}\end{aligned}.$$

It is easy to see $\left|\Phi(Q_{10}, Q_{10})\right| = 2$ and $\left|\Omega(Q_{10})\right| = 6$. If

$$\begin{aligned}&D_1' = \{\varnothing, Z_6, Z_5, Z_4, \breve{D}\},\ D_2' = \{\varnothing, Z_5, Z_6, Z_4, \breve{D}\},\ D_3' = \{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}, \\ &D_4' = \{\varnothing, Z_3, Z_6, Z_1, \breve{D}\},\ D_5' = \{\varnothing, Z_4, Z_3, Z_1, \breve{D}\},\ D_6' = \{\varnothing, Z_3, Z_4, Z_1, \breve{D}\}, \\ &D_7' = \{\varnothing, Z_6, Z_5, Z_4, Z_2\},\ D_8' = \{\varnothing, Z_5, Z_6, Z_4, Z_2\},\ D_9' = \{\varnothing, Z_6, Z_5, Z_4, Z_1\}, \\ &D_{10}' = \{\varnothing, Z_5, Z_6, Z_4, Z_1\}\end{aligned}$$

Then

$$\begin{aligned}R^*(Q_{10}) = &R(D_1')\cup R(D_2')\cup R(D_3')\cup R(D_4')\cup R(D_5')\cup R(D_6')\cup \\ &\cup R(D_7')\cup R(D_8')\cup R(D_9')\cup R(D_{10}')\end{aligned}. \quad \ldots(1)$$

(see Definition 1.9).

Lemma 1.14. Let X be a finite set, $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If by $R^*(Q_{10})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition $j)$ of the Theorem 1.3 then

$$R^*(Q_{10}) = \sum_{i=1}^{4} R(D_i') - \left|R(D_1')\cap R(D_3')\right| - \left|R(D_2')\cap R(D_4')\right| - \left|R(D_3')\cap R(D_5')\right| - \left|R(D_4')\cap R(D_6')\right|$$

(see Lemma 1.15 in [3])

Lemma 1.15. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_3(X,8)$ *and* $Z_7 = \varnothing$. *Then*

$$R^*(Q_{10}) = 10\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_4|}-4^{|\breve{D}\setminus Z_4|}\right)\cdot 5^{|X\setminus\breve{D}|}+$$
$$+10\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}+$$
$$+10\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}+$$
$$-5\cdot 2^{|Z_6\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus Z_4|}\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}-$$
$$-5\cdot 2^{|Z_3\setminus Z_4|}\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 2^{|Z_6\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}-$$
$$-5\cdot 2^{|Z_4\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}-$$
$$-5\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 2^{|Z_4\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}.$$

(see Lemma 1.16 in [3])

k′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $k)$ of the Theorem 1.3 In this case we have $\{\varnothing, Z_6, Z_5, Z_4, T, \breve{D}\}$, where $T\in\{Z_2, Z_1\}$. By definition of the semilattice D follows that $Q_{11}\vartheta_{XI} = \{\{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\}\}$.

It is easy to see $|\Phi(Q_{11}, Q_{11})| = 2$ and $|\Omega(Q_{11})| = 2$. If

$$D_1' = \{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\},\ D_2' = \{\varnothing, Z_5, Z_6, Z_4, Z_2, \breve{D}\},$$
$$D_3' = \{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\},\ D_4' = \{\varnothing, Z_5, Z_6, Z_4, Z_1, \breve{D}\}.$$

Then

$$R^*(Q_{11}) = R(D_1')\cup R(D_2')\cup R(D_3')\cup R(D_4'). \qquad \ldots(1)$$

Lemma 1.16 Let X be a finite set, $D=\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If by $R^*(Q_{11})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition $k)$ of the Theorem 1.3 then

$$|R^*(Q_{11})| = |R(D_1')|+|R(D_2')|+|R(D_3')|+|R(D_4')| =$$
$$+4\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus\breve{D}|}+$$
$$+4\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}.$$

(see Lemma 1.17 in [3])

l′) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition $l)$ of the Theorem 1.3 In this case we have $\{T, T', T'', T'\cup T'', T''', T'\cup T''\cup T'''\}$, where $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $(T'\cup T'')\setminus T'''\neq\varnothing$, $T'''\setminus(T'\cup T'')\neq\varnothing$. By definition of the semilattice D follows that

$Q_{12}\vartheta_{XI} = \{\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\}, \{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$

It is easy to see $|\Phi(Q_{12}, Q_{12})| = 1$ and $|\Omega(Q_{12})| = 4$. If

$$D_1' = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\}, D_2' = \{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\}, D_3' = \{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$$

Then

$$R^*(Q_{12}) = R(D_1')\cup R(D_2')\cup R(D_3') \quad \ldots(1)$$

Lemma 1.17 Let X be a finite set, $D=\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If by $R^*(Q_{12})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition $l)$ of the Theorem 1.3 then

$$R^*(Q_{12}) = |R(D_1')| + |R(D_2')| + |R(D_3')| - |R(D_2') \cap R(D_3')|$$

(see Lemma 1.18 in [3])

Lemma 1.18. Let $D = \{Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X,9)$ *and* $Z_7 = \varnothing$. *Then*

$$\begin{aligned} R^*(Q_{12}) = {} & 3\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot 6^{|X\setminus Z_1|}+ \\ & +3\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}+ \\ & +3\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}+ \\ & -3\cdot 2^{|Z_4\setminus(Z_6\cup Z_3)|}\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|} \end{aligned}$$

(see Lemma 1.19 in [3])

m′) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition $p)$ of the Theorem 1.3 In this case we have $\{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, By definition of the semilattice D follows that $Q_{13}\vartheta_{XI} = \{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ It is easy to see $|\Phi(Q_{13}, Q_{13})| = 1$ and $|\Omega(Q_{13})| = 1$. If $D_1' = \{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, then $R^*(Q_{13}) = R(D_1')$, $|R^*(Q_{13})| = |R(D_1')|$ and

$$|R^*(Q_{13})| = \left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 7^{|X\setminus \breve{D}|};$$

n′) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition $p)$ of the Theorem 1.3 In this case we have $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, By definition of the semilattice D follows that $Q_{14}\vartheta_{XI} = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$ It is easy to see $|\Phi(Q_{14}, Q_{14})| = 4$ and $|\Omega(Q_{14})| = 1$. If $D_1' = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$, then $R^*(Q_{14}) = R(D_1')$, $|R^*(Q_{14})| = |R(D_1')|$ and

$$|R^*(Q_{14})| = \left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(7^{|\breve{D}\setminus Z_1|}-6^{|\breve{D}\setminus Z_1|}\right)\cdot 7^{|X\setminus \breve{D}|};$$

o′) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition $p)$ of the Theorem 1.3 In this case we have $\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, By definition of the semilattice D follows that $Q_{15}\vartheta_{XI} = \{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$ It is easy to see $|\Phi(Q_{15}, Q_{15})| = 4$ and $|\Omega(Q_{15})| = 1$. If $D_1' = \{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, then $R^*(Q_{15}) = R(D_1')$, $|R^*(Q_{15})| = |R(D_1')|$ and

$$|R^*(Q_{15})| = 2\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot 4^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(5^{|Z_2\setminus Z_1|}-4^{|Z_2\setminus Z_1|}\right)\cdot\left(5^{|Z_1\setminus Z_2|}-4^{|Z_1\setminus Z_2|}\right)\cdot 7^{|X\setminus \breve{D}|};$$

p′) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition $p)$ of the Theorem 1.3 In this case we have $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, By definition of the semilattice D follows that $Q_{16}\vartheta_{XI} = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ It is easy to see $|\Phi(Q_{16}, Q_{16})| = 1$ and $|\Omega(Q_{16})| = 1$. If $D_1' = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, then $R^*(Q_{16}) = R(D_1')$, $|R^*(Q_{16})| = |R(D_1')|$ and

$$|R^*(Q_{16})| = 2\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_4|}\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(5^{|Z_2\setminus Z_1|}-4^{|Z_2\setminus Z_1|}\right)\cdot 8^{|X\setminus \breve{D}|}$$

Let us assume that

$$\begin{aligned} r_1 = {} & |R^*(Q_1)| + |R^*(Q_2)| + |R^*(Q_3)| + |R^*(Q_4)| + |R^*(Q_5)| + |R^*(Q_6)| + |R^*(Q_7)| + |R^*(Q_8)| + |R^*(Q_9)| + \\ & + |R^*(Q_{10})| + |R^*(Q_{11})| + |R^*(Q_{12})| + |R^*(Q_{13})| + |R^*(Q_{14})| + |R^*(Q_{15})| + |R^*(Q_{16})| \end{aligned}$$

Theorem 1.4. *Let* $D \in \Sigma_3(X,8)$ *and* $Z_7 = \varnothing$. *If* X *is a finite set and* R_D *is a set of all regular elements of the semigroup* $B_X(D)$. *Then* $|R_D| = r_1$.

Example 1.1. Let $X = \{1,2,3,4\}$, then $\breve{D} = \{1,2,3,4\}$, $Z_1 = \{2,3,4\}$, $Z_2 = \{1,3,4\}$, $Z_3 = \{2,4\}$, $Z_4 = \{3,4\}$, $Z_5 = \{4\}$ $Z_6 = \{3\}$, $Z_7 = \{\varnothing\}$ and we have $|R_D| = 1550$.

Reference

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Authors' address:

Faculty of physics, Mathematics and Computer Sciences, Department of Mathematics, Shota Rustaveli Batumi State University. 35 Ninoshvili St, Batumi 6010, Georgia.

E-mails: g.tavdgiridze@mail.ru
Diasamidze_ya@mail.ru