On The Diophantine Equation Of Degree Ten With Six Unknowns $3(x^2-y^2)^3+4T^6P^2(x^2-y^2)=(z^4-w^4)P^2$

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Abstract

This paper aims at determining non-zero distinct integer solutions to the Diophantine equation of degree ten with six unknowns given by $3(x^2-y^2)^3+4T^6P^2(x^2-y^2)=(z^4-w^4)P^2$. A few interesting relations among the solutions are also exhibited.

Keywords: *Non-homogeneous equation, Diophantine equation of degree ten with six unknowns, integer solutions.*

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3]. Particularly in [4,5] special equations of sixth degree with four and five unknowns are studied. In [6-8] heptic equations with three and five unknowns are analysed. In [9] an octic equation with five unknowns is considered for its integer solutions. In [10,11] octic equation with six unknowns are analysed. In [12], an octic equation with seven unknowns is considered for its distinct integer solutions. This paper concerns with the problem of determining non-trivial integral solution of the non-homogeneous equation of the tenth degree with six unknowns given by

$$3(x^2-y^2)^3+4T^6P^2(x^2-y^2)=(z^4-w^4)P^2.$$

A few relations between the solutions and the special numbers are presented.

2. Method of Analysis:

The Diophantine equation of degree ten with six unknowns to be solved is given by

$$3(x^2-y^2)^3+4T^6P^2(x^2-y^2)=(z^4-w^4)P^2 \qquad (1)$$

Introducing the linear transformations

$$\left.\begin{array}{l} x=u+v \\ y=u-v \\ z=u+2v \\ w=u-2v \\ P=2u \end{array}\right\} \qquad (2)$$

in (1), it simplifies to

$$u^2+v^2=T^6=(T^3)^2 \qquad (3)$$

which is in the form the well known Pythagorean equation.Employing the most citied solution of the Pythagorean equation, the values of u, v & T are given by

$$u=2pq,\ v=p^2-q^2, p>q>0 \qquad (4)$$

$$T^3=p^2+q^2 \qquad (5)$$

Note that , (5) is satisfied by

$$\begin{array}{l} p=m(m^2+n^2) \\ q=n(m^2+n^2) \\ T=m^2+n^2, m>n \end{array} \qquad (6)$$

$$\therefore \begin{array}{l} u=2mn(m^2+n^2)^2 \\ v=(m^2-n^2)(m^2+n^2)^2 \end{array}$$

Substituting the above values of u, v in (2), we have

$$\left.\begin{array}{l} x = x(m,n) = (m^2+n^2)^2(2mn+m^2-n^2) \\ y = y(m,n) = (m^2+n^2)^2(2mn-m^2+n^2) \\ z = z(m,n) = (m^2+n^2)^2(2mn+2m^2-2n^2) \\ w = w(m,n) = (m^2+n^2)^2(2mn-2m^2+2n^2) \\ P = P(m,n) = 4mn(m^2+n^2)^2 \end{array}\right\} \quad (7)$$

Thus, (6) & (7) represent the required non-zero integer solutions to (1).

A few interesting relations among the solutions are presented below:

1. $P(6n,n)$ is a Nasty number.
2. $4(xy - zw) = 12T^6 - 3P^2$
3. $P^2 = 2(xy + T^6)$
4. $5P^2 = 4zw + 16T^6$
5. $7P^2 = 4(xy + zw + 5T^6)$

Note that (5) is also satisfied by

$$p = m^3 - 3mn^2$$
$$q = 3m^2n - n^3$$
$$T = m^2 + n^2$$

For this choice, the corresponding non-zero distinct integer solutions to (1) are given by

$$x = x(m,n) = f^2(m,n) - g^2(m,n) + 2f(m,n)g(m,n)$$
$$y = y(m,n) = 2f(m,n)g(m,n) - f^2(m,n) + g^2(m,n)$$
$$z = z(m,n) = 2f(m,n)g(m,n) + 2f^2(m,n) - 2g^2(m,n)$$
$$w = w(m,n) = 2f(m,n)g(m,n) - 2f^2(m,n) + 2g^2(m,n)$$
$$P = P(m,n) = 4f(m,n)g(m,n)$$
$$T = T(m,n) = m^2 + n^2$$

where,

$$f(m,n) = m^3 - 3mn^2 \ \& \ g(m,n) = 3m^2n - n^3, \forall m \geq 4n$$

A few interesting relations among the solution are exhibited below:

1. $12T^6 = 4(xy - zw) + 3P^2$
2. $7P^2 = 4(xy + zw + 5T^6)$
3. $P^2 = 2(xy + T^6)$
4. $5P^2 = 4zw + 16T^6$
5. $2x^2 - 2xP + P^2 = 2T^6$
6. $P^2 + 2y^2 - 2Py = 2T^6$

It is worth mentioning here that, in the linear transformations given by (2), the values of z and w may be taken as

$$z = 2u + v \quad \& \quad w = 2u - v$$

For this choice, (1) simplifies to

$$4u^2 = 2v^2 + T^6$$

which is satisfied by

$$\begin{array}{l} u = u(\alpha,\beta) = 4\alpha^2 f^2(\alpha,\beta) + 8\beta^2 g^2(\alpha,\beta) \\ v = v(\alpha,\beta) = 16\alpha\beta f(\alpha,\beta)g(\alpha,\beta) \\ T = T(\alpha,\beta) = 2\alpha^2 - 4\beta^2 \end{array} \quad (8)$$

where, $f(\alpha,\beta) = \alpha^2 + 6\beta^2$, $g(\alpha,\beta) = 3\alpha^2 + 2\beta^2$

Substituting the above values of u, v in (2), we have

$$\left.\begin{array}{l} x = x(\alpha,\beta) = 4\alpha^2 f^2(\alpha,\beta) + 8\beta^2 g^2(\alpha,\beta) + 16\alpha\beta f(\alpha,\beta)g(\alpha,\beta) \\ y = y(\alpha,\beta) = 4\alpha^2 f^2(\alpha,\beta) + 8\beta^2 g^2(\alpha,\beta) - 16\alpha\beta f(\alpha,\beta)g(\alpha,\beta) \\ z = z(\alpha,\beta) = 8\alpha^2 f^2(\alpha,\beta) + 16\beta^2 g^2(\alpha,\beta) + 16\alpha\beta f(\alpha,\beta)g(\alpha,\beta) \\ w = w(\alpha,\beta) = 8\alpha^2 f^2(\alpha,\beta) + 16\beta^2 g^2(\alpha,\beta) - 16\alpha\beta f(\alpha,\beta)g(\alpha,\beta) \\ P = P(\alpha,\beta) = 8\alpha^2 f^2(\alpha,\beta) + 16\beta^2 g^2(\alpha,\beta) \end{array}\right\} (9)$$

Thus, (8) & (9) represent the required non-zero integer solutions to (1).

3. Conclusion:

In conclusion, one may search for different patterns of solutions to (1) and their corresponding relations.

References

[1] Dickson L.E., History of Theory of Numbers, Vol-11, Chelsea Publishing company, New York (1952).

[2].Mordel L.J., Diophantine equations, Academic Press, London (1969).

[3].Carmichael R.D., The theory of numbers and Diophantine Analysis, Dover Publications, New York (1959)

[4]. Gopalan M.A, Vidhyalakshmi S, and Lakshmi K,On the non homogeneous sextic equation $X^4 + 2(X^2 + w)X^2Y^2 + Y^4 = Z^4$, *IJAMA*, 4(2),171- 173,Nov-2012.

[5].Gopalan M.A, Vidhyalakshmi S, and Lakshmi K., Integral Solution of the sextic

equation with five unknowns
$x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5$, IJESRT, 502-504, Dec- 2012.

[6].Gopalan M.A, and Sangeetha G., Parametirc integral solutions of the heptic equation with five unknowns
$x^4 - y^4 + 2(x^3 + y^3)(x-y) = 2(X^2 - Y^2)z^5$
Bessel Journal of Mathematics 1(1), 17- 22, 2011.

[7]. Gopalan M.A, and Sangeetha G., On the heptic Diophantine equations with five unknowns $x^4 - y^4 = (X^2 - Y^2)z^5$, Antarctica Journal of Mathematics, 9(5), 371-375, 2012.

[8]. Manjusomnath, Sangeetha G, and Goapaln M.A., On the non-homogeneous heptic equations with three unknowns $x^3 + (2^p - 1)y^5 = z^7$, Diophantous journal of Mathematics, 1(2),117-121, 2012.

[9]. Vidhyalakshmi S, Lakshmij K, Gopalan M.A, Obervation on the non-homogeneous equation of the eighth degree with five unknowns $x^4 - y^4 = (k^2 + s^2)(z^2 - w^2)p^6$, IJIRSET, Vol-2(5), 1789-1798.

[10]. Vidhyalakshmi S, Lakshmij K, Gopalan M.A, On the non-homogeneous equation of the eighth degree with six unknowns $x^5 - y^5 + (x^3 - y^3)xy = p(z^2 - w^2)^2 T^3$ IJESRT, Vol-2(5),1218-1223, May-2013.

[11]. Vidhyalakshmi S, Lakshmij K, Gopalan M.A, Observations on the non- homogeneous equation of the eighth degree with six unknowns $x^6 - y^6 - 2z^3 = (w^2 - p^2)T^6$. The international Journal of science & Technolodge, Vol-2(3), 14-22, March- 2014.

[12]. Vidhyalakshmi S,Kavitha A, Gopalan M.A, Integer solutions of the eighth degree with seven variables
$(x^2 - y^2)(4x^2 + 4y^2 - 6xy) =$
$2(w+p)(T+s)z^6$
International Journal of Mathematical Archive -6(8), 1-4, August -2015.