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A θ-closed Graph Theorem

A. T. Al-Ani

College of Science and Information Technology, Department Of Mathematics
Irbid National University, Irbid, Jordan

Abstract: We prove a θ-closed graph theorem using mH-closed spaces, where m is an infinite cardinal number.

1- Introduction and Preliminaries

In this paper m is infinite cardinal number. A topological space X is quasi mH-closed space iff every open cover (equivalently, regular open cover) of X with cardinality at most m has a finite subcollection the closures of its members cover X. . A Hausdorff space is H-closed iff it is closed in every Hausdorff space it can be embedded. A Hausdorff space is quasi mH-closed iff it is closed in every space with character m. A space is of character m iff every point has a local base of cardinality less than or equal to m.
Theorems 2.1 and 2.2 modify several characterizations of H-closed spaces in [2] and [4] to characterizations of mH-closed spaces.

A subset A of a space X is called regular open iff $A = \bar{A}^{\,o}$. A subset A is regular closed iff its complement is regular open(i.e $A = \overline{A^o}$).

A filterbase $\Im=\{F_\lambda:\lambda\in\Lambda\}$ in a space X is said to r-accumulate [4] to $x_o\in X$ iff for each $F_\lambda\in\Im$ and each open set (equivalently [3], each regular open set) V containing x_o we have $F_\lambda \cap \bar{V} \neq \varphi$.

A function f: X→ Y has a strongly – closed graph if for each
(x, y)∈G(f), the graph of f, there exists open sets U in X and V in Y containing x and y respectively, such that $(U \times \bar{V}) \cap G(f) \neq \varphi$.

A function f: X→Y is called mwc-weakly continuous, mwc if for each x∈X and for each open set V in Y such that the cardinality of Y\V is ≤m there exists an open set W in X containing x such that $f(W)\subset\bar{V}^o$.

A multifunction α of a topological space Y is a set valued function such that α(x)≠ϕ for every x in X. A multifunction α is called closed graph iff its graph {(x,y):y∈α(x)} is closed in X×Y.

A subset A of a space X is called quasi mH-closed relative to X iff, for each cover of A by open subsets of X, of cardinality ≤ m there is a finite subcollection the closures of its members cover A. If X is quasi mH-closed relative to X we say, simply, that X is quasi mH-closed. A Hausdorff quasi mH-closed space is mH-closed.
Theorem 3.2 characterizes quasi mH-closed set relative to a space X.

A subset A of X is called θ-closed iff for every x∈X\A there is an open set V in X such that x∈V and $\bar{V} \cap A = \varphi$.
A multifunction α of X into Y is called θ-closed graph iff its graph is θ-closed in X×Y.
Theorem 3.5 characterizes quasi mH-closed spaces in terms of θ-closed graph multifunctions. Corollary 3.6 gives a new characterization of quasi H-closed spaces .

2- mH-closed space

In this section several characterizations of mH-closed spaces are given. They are modifications of characterizations of H-closed spaces appeared in [2] and [4].

Theorem 2.1 The following are equivalent about a Huasdorff space X.

(i) X is mH-closed.
(ii) Every filterbase of cardinality ≤ m, of open sets has a cluster point.
(iii) Every family of cardinality ≤ m of closed sets of X whose intersection is empty has a finite subfamily with interiors has empty intersection .
(iv) Every open cover of cardinality ≤ m of X has a finite subfamily the closures of its members cover X.

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Theorem 2.2 Let X be a Hausdorff space. Then the following are equivalent.

(i) X is mH-closed.

(ii)For each family of regular-closed sets ℑ={Fλ:λ∈ Λ} of cardinality ≤m

such that $\underset{\lambda\in\Lambda}{intersect} F_\lambda = \varphi$, there exists a finite subfamily {Fλi: i=1,2,…,k} such $\underset{i=1}{\overset{k}{intersect}} {F_{\lambda i}}^{o} = \varphi$.

(iii)Each filterbase ℑ={F_λ: λ∈ Λ} of cardinality ≤ m r-accumulates to some point x_0∈ X.

Proof (i)⇒(ii) Follows from the previous theorem because ℑ is a family of closed sets with cardinality ≤ m.

(ii)⇒(i) Let{V_λ:λ∈ Λ} be a cover of X by regular open sets in X of cardinality ≤ m. Then $\underset{\lambda\in\Lambda}{} V_\lambda = X$. So that $\underset{\lambda\in\Lambda}{} {V_\lambda}^c = \varphi$ Since each V_λ^c is regular-closed by hypothesis there is a finite subfamily {$V_{\lambda i}^c$: i= 1, 2, …, k} such that $\underset{i=1}{\overset{k}{intersect}} V_{\lambda i}^{c^o} = \varphi$.

But $V_{\lambda i}^{c^o} = \bar{V}_{\lambda i}^c$,So

$\underset{i=1}{\overset{k}{intersect}} \bar{V}_{\lambda i}^c = \varphi$Thus $\underset{i=1}{\overset{k}{}} \bar{V}_{\lambda_i} = X$and so X is mH-closed.

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(i)⟹(iii) Suppose that there exists a filterbase $\Im=\{F_\lambda:\lambda\in \Lambda\}$ in X of cardinality ≤ m that does not r-accumulate in X.

Then for each x∈X, ∃ an open set V(x) containing x and $F_{\lambda(x)}\in\Im$ such that $F_{\lambda(x)} \cap \overline{V_{(x)}} = \varphi$. Let $W_\lambda(x) = {}_{x\in X} V(x)$such that $F_\lambda(x) \cap \overline{V(x)} = \varphi$. Then $\{W_{\lambda(x)}: \lambda\in \Lambda\}$is an open cover of X with cardinality ≤m. Since X is mH-closed there is a finite subfamily $\{W_{\lambda(x1)}, W_{\lambda(x2)},\ldots, W_{\lambda(xk)}\}$such that

$$\overset{k}{\underset{i=1}{}}\overline{W_{\lambda(\text{xi})}} = X.$$

Since $\Im$ is a filterbase on X, there exists $F_{\lambda o}\in\Im$, such that

$F_{\lambda o} \subset \overline{\underset{i=1}{\overset{k}{intersect}} F_\lambda}(\text{xi})$. Then $F_{\lambda o} \cap \overline{W_\lambda(\text{xi})} \neq \varphi$ for some i.

(Because $\overline{W_\lambda(\text{xi})}$is a cover of X). So that $F_{\lambda o} \cap \overline{V_\lambda(\text{xi})} \neq \varphi$. Then $F_{\lambda o} \cap \overline{V_\lambda(\text{xi}_o)} \neq \varphi$for some i_o=1,2,…,k. Consequently $F_{\lambda o} \cap \overline{V}_\lambda(\text{xi}_o) \neq \varphi$which is a contradiction. Thus **(i)⟹(iii).**

(iii)⟹(ii) Let $\Im=\{F_\lambda:\lambda\in\Lambda\}$ be a family of cardinality ≤ m of regular closed and suppose that each finite subcollection $\{F_{\lambda i}$: i=1,2,…,k} with

$$\underset{i=1}{\overset{k}{intersect}} \overset{o}{F_{\lambda i}} \varphi$$

We shall prove that $\underset{\lambda\in\Lambda}{intersect} F_\lambda \varphi$.

$\{F_\lambda^o\}$is a filterbase of open sets of cardinality ≤ m on X. Then by hypothesis there is a point $x_o\in$X such that

$F_\lambda^o \overline{V} \neq \varphi$,

for every λ∈Λ and every open set V containing x_o .

Then

$F_\lambda \cap \overline{V} \neq \varphi$ for each λ∈Λ,

because $F_\lambda^o F_\lambda$.

If $\underset{\lambda\in\Lambda}{intersect} F_\lambda = \varphi$then $x_o \notin \underset{\lambda\in\Lambda}{intersect} F_\lambda$So there is β∈Λ such that $x_o \notin F_\beta$. So, $F_\beta^c$is regular open set containing x_o with $F_\beta^{\bar{c}} \cap F^0_\beta = \phi$.

This means that$\Im$ does not r-accumulate to x_o . Contradiction

Theorem 2.3 Let Y be an mH-closed space. Then for every space X , every strongly-closed graph function f: X →Y is mwc.

Proof Let x∈X and let V be a regular open set in Y such that V^c has cardinality ≤ m and f(x) ∈V. Let y∈V^c . Then (x, y)∉G(f) so there exist open sets $U_y(x) \subset$ X and V(y) ⊂ Y containing x and y respectively such that $[U_y(x) \times \overline{V(y)}] \cap G(f) = \phi$. Since Y is Hausdorff we can choose V(y) such that f(x)∉ $\overline{V(y)}$. Now {V(y): y∈V^c } is an open cover of the regular closed set V^c with cardinality ≤ m. And since V^c is mH-closed there is a finite subcollection $\{V(y_1), V(y_2), \ldots,V(y_k)\}$ such that $\{\overline{V(y_1)}, \overline{V(y_2)}\ldots, \overline{V(y_k)}\}$covers V^c. Let

$W = \underset{i=1}{\overset{k}{intersect}} U_{\text{yi}}(x)$, then W is open in X such that f(W) is disjoint from $\overset{k}{\underset{i=1}{}}\overline{V(y_\iota)}$

So, f(W)⊂V⊂$\overline{V}$. It follows that f is mwc at x.

3- Quasi mH-closed space

Our main result here is Corollary 3.6 a characterizations of quasi mH-closed spaces in terms θ-closed graph multifunctions. Theorem 3.3 is a generalization of a result about compact spaces.

Theorem 3.1 A space X is quasi mH-closed iff for every regular open cover$\{V_\lambda : \lambda\in\Lambda\}$of X, of cardinality ≤ m , there is a finite subfamily $\{V_{\lambda_1}, V_{\lambda_2},...., V_{\lambda_k}\}$ such that $\bigcup_{i=1}^{k}\bar{V}_{\lambda_i}$is a cover of X.

Proof If X is mH-closed then a regular open cover is an open cover and so it satisfies the condition in the statement of the theorem.
Conversely suppose that X satisfies the condition. Let $\{V_\lambda :\lambda\in\Lambda\}$ be an open cover of X with cardinality ≤m.
Then $\bar{V_\lambda}^{o}$is regular open for every $\lambda\in\Lambda$ and $V_\lambda = V_\lambda^o \subset \bar{V_\lambda}^{o}$. So $\{\bar{V_\lambda}^{o} :\lambda\in\Lambda\}$ is a regular open cover of X satisfying the conditions. So it has a finite subfamily $\{\overline{V_{\lambda_1}}^{o}, \overline{V_{\lambda_2}}^{o},...., \overline{V_{\lambda_k}}^{o}\}$such that $\bigcup_{i=1}^{k}\overline{\overline{V_{\lambda_i}}^{o}}$covers X. But $\overline{\overline{V_{\lambda_i}}^{o}} \subset \overline{V_{\lambda_i}}$ for all i=1,2,...,k. So, $\bigcup_{i=1}^{k}\overline{V_{\lambda_i}}$ is a cover of X. Thus X is mH-closed.

Theorem 3.2 A subset K of a space X is quasi mH-closed relative to X if and only if for each filterbase Ω on X with cardinality at most m such that $F\cap C\neq\phi$ is satisfied for each $F\in\Omega$ and C regular closed set containing K we have $K\cap ad_\theta\, \Omega \neq \phi$.

Proof Suppose that the condition is satisfied . We shall prove that K is quasi mH-closed relative to X. Suppose , on the contrary, that K is not quasi mH-closed relative to X. Then there is a cover θ of K of cardinality ≤ m by open sets in X such that no finite $\theta^*\subset\theta$ satisfy $K\subset \bigcup_{V\in\theta}\bar{V}$. This means that $K \cap intersect_{V\in\theta}\bar{V}^c \neq \varphi$. So $\Omega = \{\bar{V}^c: V \in \theta\}$ is a filterbase on X satisfying $\bar{V}^c \cap K \neq \varphi$for every $\bar{V}^c\in\Omega$. Thus by hypothesis we have $K\cap ad_\theta\Omega\neq\phi$. That is, $K\cap intersect_{V\in\theta}\bar{V}^{\bar{c}} \neq \varphi$.

Then $K \not\subset \bigcup_{v\in\theta}\bar{V}^o$. But $\bar{V}^{o} \supset V$.
So we get $K \not\subset \bigcup_{v\in\theta} V$ Hence θ is not a cover of K. Contradiction. Thus K is quasi mH-closed relative to X.

Now, suppose that $K\cap ad_\theta\Omega = \phi$ for some filterbase Ω of cardinality at most m. Then for each $x\in K$ there is V(x) open containing x and $F(x)\in\Omega$ with $\overline{V(x)} \cap F(x) = \varphi$. Let $W_x = \{V(y): \overline{V(y)} \cap F(x) = \varphi\}$. Then $\{W_x: x\in K\}$ is an open cover of K with cardinality ≤ m . Since K is quasi mH-closed there is a finite set $K^*\subset K$ with $K \subset \bigcup_{x\in K}\bar{W}_x$.Choose $F^*\in\Omega$ with $F \subset intersect_{x\in K}F(x)$. Then $F \cap \bigcup_{x\in K}\overline{W_x} = \varphi$.

The following result is a modification of a result about compact spaces to m-compact spaces.

Theorem 3.3 If X is m-compact T_1 space with character m then it has a base of cardinality ≤ m

Proof For each x∈X let β_x be a local base at x of cardinality ≤ m. Let B=∪{B_x : x∈X} . Then B is a base for X. We shall prove that B has cardinality ≤ m . Let S be the collection of all minimal open covers (open covers having no strictly subcovers) of cardinality ≤ m . Since X is
m-compact T_1 every member of S is a finite subcover. And |B|=|B| |S|. The rest of the proof is as in [1] (page 178 , problem 120).

Theorem 3.4 A regular closed subset of a quasi mH-closed space is quasi mH-closed.

Proof Let A be a regular closed set then $A = \bar{U}$ for some open set U.
Let {V_λ :λ ∈Λ } be a filterbase on A with cardinality ≤ m. Then
{V_λ ∩ U : λ∈Λ} is an open filterbase on X with cardinality ≤ m. So it has a cluster point. This cluster point belongs to A. So A is mH-closed.

Theorem 3.5 A space X is quasi mH-closed iff every θ-closed graph multifunction of X to a space Y with character m maps regular closed sets in X onto θ-closed sets, in Y.

Proof Let X be a quasi mH-closed space. Y be a space of character m and α has a θ-closed graph multifunction of X to Y. Let K be a regular closed subset of X and z∈∈ cl_θ (α(K)). Let Ω be a local base at z of cardinality at most m.
Then α^{-1} (Ω) is a filterbase on X with cardinality at most m such that F∩K≠ϕ for every Fεα^{-1}(Ω) . And since K is regular closed it follows that it is quasi mH-closed .
Hence K∩$ad_\theta\alpha^{-1}$(Ω)≠ϕ. Thus for any x∈K∩$ad_\theta\alpha^{-1}$(Ω) we have $\bar{V}$∩α^{-1}(W)≠ϕ for every open set V containing x and W∈Ω. Consequently ($\bar{V}$×W) ∩G(α) ≠ϕ. So (x, z)∈cl_θ(G(α))=G(α) and hence z ∈α(x).
Conversely. Let Ω be a filterbase on X of cardinality at most m. Let a∉X, and Y=X∪ {a}, Topologize Y by taking every point in X open inY and a set containing a be open in Y iff it contains a member of Ω. Let α be the θ-closure of the identity function of X into Y. Then a ∈cl_θ(α(X)). Since α(X) is regular closed it follows that there is x in X such that a ∈α(x). This x must belong to ad_θΩ. Thus X is quasi mH-closed.

The following result is a new characterization of quasi H-closed spaces.

Corollary 3.6 A space X is quasi H-closed space iff every θ-closed graph multifunction of X to a space Y, maps regular closed sets in X onto θ-closed sets in Y.

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