Generalization of Steckly Parameter and Biot Number: Case of 3D superconductor.

S. Saoud [1]; M. El Khomssi [2]; Z.Mahani[3].

ABSTRACT

Most work related to the thermal stability of superconductors maintained by cryogenics, consider the domain as a filiform with an infinite length. This description consists to study an unidimensional domain, which characterized by some fundamental installation parameters, particularly Steckly Parameter and Biot Number [1][2][3][4][6][8] [9]. In this paper, we will introduce average values of the thermophysical coefficients, and we will use compacity of the domain for obtaining the following:

- Mathematical equation given the thermic feild modelling the physical state of a three- dimensional superconductor.
- Generalization of Steckly Parameter and Biot Nomber for the three-dimensional case.
- Bifurcation parameter, which help in analysis of existence and uniqueness.

Keywords: Mathematical modeling, Superconductor, Heat equation, Steckly Parameter, Biot Number.

1. Equation governing the thermal state of a superconducting

1.1. Critical values and superconductor domain.

The superconducting state is governed by three quantities: the temperature T, the magnetic field H and the electrical current I. This state exists if and only if the triplet $(T, \|\overrightarrow{H}\|, I)$ belongs to a tetrahedron domain:

$$D = \{(T, H, I), \quad T \leq T_c, \ I \leq I_c, \ H \leq H_c\}$$

So, to preserve the superconducting state, it is indispensable that for a fixed magnetic field $\overrightarrow{H}$ (of modulus less than the critical one H_c), the temperature and the electrical current cannot exceed respectively the thresholds T_c and I_c called critical temperature and critical electric current. The materials used for the construction of superconducting electromagnets are only the ones whose critical temperature is very low, cooled by the liquid Helium. To enable the transport of large current in high magnetic field, the superconducting wires consist of a number of fine type -II superconducting filaments (Niobium Titan (N_bT_i) or Niobium Etan (Nb_3S_n)) twisted inside a matrix of normal metal , generally pure copper. In many applications,the conductor are complex and include other materials.

1.2. Energy Balance. The energy transfer takes place whenever a temperature gradient exists within a system, or when two systems are in contact with different temperatures.

[1]ERTISED, University Ibn Zohr, Agadir, Morocco
[2]CSSI, Universty Sidi Mohamed Ben Abdellah, Fes, Morocco
[3]LASIME Universty Ibn Zohr, Agadir, Morocco

In superconductivity, the energy conservation in a physical volume Ω_s delimited by a boundary $\partial\Omega_s$ is in the form[2][3]:

$$\frac{\partial}{\partial t'}\int\int\int_{\Omega_s} E dv = -\int\int\int_{\Omega_s} div(q)dv + \int\int\int_{\Omega_s} W dv + \int\int\int_{\Omega_s} P dv \qquad (1)$$

The left member derives from the internal energy, while the first term of the right member represents the conduction flow heat on $\partial\Omega_s$. The quantity W is a power created within the volume Ω_s, and P is any thermic perturbation.
The Fourrier hypothesis linking the flow density $\overrightarrow{q}$ to the local temperature at t' gives that:

$$\overrightarrow{q}(T) = -\overline{\overline{K}}(T)grad(T) \qquad (2)$$

where $\overline{\overline{K}}$ is the thermic conductivity tensor whose the elements $(K_{ij}(T))_{ij}$ are the components of the thermic conductivity matrix associated to the superconductor.
It's known that the application of the first and second principles of thermodynamics, for a continuous field, is reduced in the absence of the substance transfer in the heat equation:[3][5]:

$$C(T)\frac{\partial T}{\partial t'} = div(\bar{\bar{K}}(T)grad(T)) + W(T) + P(X,\ t') \qquad (3)$$

In fact W represents the energetic competition between the power dissipated by Joule effect G and that transferred by the cryogenic bath Q:

$$W = G - Q \qquad (4)$$

On the one hand, the physical nature of the term G requires it to be a function that varies on temperature, on the other hand Q depends explicitly on the cooling system. For our study, the temperature is imposed at any point in the boundary of Ω_s^4 , therefore if we suppose that the bath temperature is T_b, the boundary conditions are Dirichlet type, and we obtain:

$$T(X,t') = T_b \quad in \quad \partial\Omega_s \quad et \quad t' \geq 0 \qquad (5)$$

The initial data is:

$$T(X,0) = T_0(X) \quad for \quad X \in \Omega_s \quad et \quad t' = 0 \qquad (6)$$

2. Dimensional analysis

2.1. Reduced quantities. To facilitate our study, we will establish a dimensional analysis. The fundamental quantities of the problem are :

- T_c : Critical temperature of superconductor.
- L_i : Maximum length following the direction e_i of Ω_s.
- T_b: Cryogenic bath temperature.
- I_c : Critical Intensity of material.

We introduce here the following reduced quantities:

$$T^* = T_c - T_b, \quad T = T^* u + T_b, \quad X_i = L_i x_i, \quad I = iI_c$$

We consider τ as an arbitrary characteristic time linking the physical time and the abstract time by the report:

$$\tau = \frac{t'}{t}$$

The field u is the new unknown of the problem, in fact it's a reduced thermic field.
We recall that Ω_s is a physical domain superconductor considered as a widely bounded regular feild. Let B as follow:

$$X = Bx$$

In other words, B is the matrix defined by:

$$B = (\delta_{ij} L_i)$$

where δ_{ij} is Kronecker symbol. In this stage, B gives that Ω_s can be write as:

$$\Omega = \{x \in IR^N / \ x = B^{-1}X \ for \ X \ in \ \Omega_s\} = B(\Omega_s)$$

In order to rewrite the problem according to u, for all function ψ having a physical variables, we introduce the function $\widetilde{\psi}$ as follow:

$$\widetilde{\psi}(x,t) = \psi(\tilde{X}, t') \quad and \quad (x,t) = \begin{pmatrix} B^{-1} & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} X \\ t' \end{pmatrix}$$

2.2. Average values of thermophysical coefficients. Let C^* be the average value of the specific heat C on Ω given by:

$$C^* = \frac{1}{\mid \Omega \mid} \int_\Omega \widetilde{C}(\omega) d\omega \tag{7}$$

where $\mid \Omega \mid$ is the measure of Ω. We put that:

$$c(u) = \frac{\widetilde{C}(\omega)}{C^*}$$

and we obtain the following equation:

$$C(T)\frac{\partial T}{\partial t'} = \Big(\frac{C^* T^*}{\tau}\Big) c(u) \frac{\partial u}{\partial t} \tag{8}$$

It's known that thermic conductivity tensor is written in the form:

$$\overline{\overline{K}}(T) = \begin{pmatrix} K_{11}(T) & K_{21}(T) & K_{13}(T) \\ K_{21}(T) & K_{22}(T) & K_{23}(T) \\ K_{31}(T) & K_{32}(T) & K_{33}(T) \end{pmatrix}$$

Let K^*_{ij} be constants defined by:

$$K^*_{ij} = \frac{1}{|\Omega|} \int_\Omega \widetilde{K}_{ij}(u) du$$

The new components of the tensor are:

$$k_{ij}(u) = \frac{\widetilde{K}_{ij}(u)}{\varepsilon_{ij} K^*}$$

where ε_{ij} are an arbitrary parameters which will be chosen later. There, we have the following:

$$\frac{\partial}{\partial X_i}\Big(K_{ij}(T)\frac{\partial T}{\partial X_j}\Big) = \Big(\frac{\varepsilon_{ij} K^* T^*}{L_i L_j}\Big)\frac{\partial}{\partial x_i}\Big(k_{ij}(u)\frac{\partial u}{\partial x_i}\Big) \tag{9}$$

2.3. Normalisation and bifurcation parameter. For raisons of dimensional analysis, we introduce a new quantity which will be compatible with the nature of W. There, we consider W^* :

- When W is finished for any temperature field T, we put:

$$W^* = sup_{T \geq 0} W(T) < +\infty$$

- If W is not bounded, we choose:

$$W^* = sup_{T_2 \geq T \geq 0} W(T)$$

From this choice, we take $F(u)$ as follow:

$$F(u) = \frac{\widetilde{W}(u)}{W^*} \tag{10}$$

Physical nature of P gives that:

$$P(X, 0) = 0 \quad et \quad | P(X, t') | \leq max_{\Omega \times IR^+} P(X, t') = P_\infty$$

Practically, the disturbance P can always improve with time. Mathematically, we describe this situation by:

$$There\ is\ \ t^* > 0\ \ such\ as\ for\ all\ t \geq t^*\ \ p(X, t) = 0$$

We consider the function P:

$$P(x, t) = \frac{P(X, t')}{P_\infty} \leq 1 \tag{11}$$

From (8), (9), (10) and (11), the thermic state of a superconductor filed, is given by the following mathematic model:

$$\Big(\frac{C^* T^*}{\tau}\Big)c(u)\frac{\partial u}{\partial t} + \sum_{i,j=1}^{3}\Big(\frac{\varepsilon_{ij} K^* T^*}{L_i L_j}\Big)\frac{\partial}{\partial x_i}\Big(k_{ij}(u)\frac{\partial u}{\partial x_i}\Big) = W^* F(u) + P_\infty P(x, t) \tag{12}$$

This modeling represents a ameliorate version of [3][4]. We define the domain compactness μ as:

$$\mu = \frac{A}{V}$$

where A is the exchange surface and V the domain volume. Let L^* be the quantity given by:

$$L^* = \Big(sup_{i=1}^{3} \prod_{i=1}^{3} L_i\Big)\mu \qquad (13)$$

We multiply equation (12) by term of equation (13), we obtain:

$$\Big(\frac{C^* L^*}{\tau K^*}\Big) c(u)\frac{\partial u}{\partial t} + \sum_{i,j=1}^{3} \Big(\frac{\varepsilon_{ij} K^* L^*}{L_i L_j}\Big)\frac{\partial}{\partial x_i}\Big(k_{ij}(u)\frac{\partial u}{\partial x_i}\Big) = \frac{W^* L^*}{T^* K^*} F(u) + \frac{L^* P_\infty}{T^* K^*} P(x,t)$$

What we have introduced as floating parameters, we choose it now explicitly :

$$a = \frac{L^* P_\infty}{T^* K^*};\quad \lambda = \frac{W^* L^*}{T^* K^*};\quad \tau = \frac{C^* L^*}{K^*};\quad \varepsilon_{ij} = \frac{L_i L_j}{L^*} \qquad (14)$$

For the initial data, we take the field u_0 as:

$$u(x,0) = u_0(x) = \frac{T_0(X) - T_b}{T^*} \qquad (15)$$

We have $\partial\Omega = B(\partial\Omega_s)$, which give that the boundary condition:

$$u(x,t) = 0 \quad for \quad all \quad (x,t) \ \in \partial\Omega \times IR^+ \qquad (16)$$

3. Problems modeling the dynamic and the stationary thermal state.

3.1. Dynamic Model and general data . From the choice takin in (14), and equations (12), (14), (15) and (16), the problem managing the thermal state of a three-dimensional superconductor admits the following mathematical formulation:

$$(P_e)\begin{cases} c(u)\frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(k_{ij}(u)\frac{\partial u}{\partial x_i}\Big) = \lambda F(u) + aP(x,t) \quad on \quad \Omega \times IR^+ \\ u(x,t) = 0 \quad on \quad \partial\Omega \times IR^+ \\ u(x,0) = u_0(x) \quad on \quad \Omega \times \{0\} \end{cases}$$

For a physical raisons, solution of this problem must bellow to the space:

$$E = C(IR^+, C^0(\Omega)) \cap \ C(IR^+, H_0^1(\Omega)) \cap \ C^1(IR^+, L^2(\Omega))$$

3.2. Stationary problem. The states of balance are solutions of the stationary problem:

$$(P_s)\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(k_{ij}(u)\frac{\partial u}{\partial x_i}\Big) = \lambda F(u) \quad on \quad \Omega \\ u = 0 \quad on \quad \partial\Omega \end{cases}$$

For this problem, we search a solution in the space $H_0^1(\Omega)$.

3.3. Equivalent problem: homogeneous isotropic case. In this part, we consider that the field is an homogeneous isotropic domain [7][9]. This assumption gives that the thermic conductivity tensor can be written in the following form:

$$K(u) = k(u)\delta_{ij}, \quad where \ \ \delta_{ii} = 1 \ \ and \ \delta_{ij} = 0 \ \ for \ \ i \neq j \tag{17}$$

The function $k(u$ is strictly positive and increasing of $C^1(\Omega \times IR^+)$. In this case, we obtain:

$$(\widetilde{P_e}) \begin{cases} c(u)\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(k(u)\frac{\partial u}{\partial x_i}\right) = \lambda F(u) + aP(x,t) \quad \text{on} \quad \Omega \times IR^+ \\ u(x,t) = 0 \ \ on \ \ \partial\Omega \times IR^+ \\ u(x,0) = u_0(x) \quad on \ \ \Omega \times \{0\} \end{cases}$$

The stationary problem corresponding is:

$$(\widetilde{P_s}) \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(k(u)\frac{\partial u}{\partial x_i}\right) = \lambda F(u) \ \ on \ \ \Omega \\ u = 0 \ \ on \ \ \partial\Omega \end{cases}$$

A large part of thermal stability analysis[3][4][6] is based on the nature and the number of the possible stationary solutions, it seems interesting to transform the differential operator of $(\widetilde{P_s})$. This is possible due to the Kirchoff transformation:

$$y = Y(u) = \int_0^u k(s)ds \tag{18}$$

Then, we have:

$$k(u)\frac{\partial u}{\partial x_i} = \frac{\partial y}{\partial x_i}$$

If we use the fact that Y is a bijection [2][3][4], boundary condition remains invariant:

$$u(x) = 0 \quad on \ \ \partial\Omega \Longleftrightarrow y(x) = 0 \ \ on \ \ \partial\Omega$$

Finally, we obtain the equivalent problem $(\widetilde{P_s}')$:

$$(\widetilde{P_s}') \begin{cases} -\Delta y = \lambda F(u) \ \ on \ \ \Omega \\ y(x) = 0 \ \ on \ \ \partial\Omega \end{cases}$$

4. Extension of Steckly Parameter and Biot Number.

4.1. Recall for the one-dimension case. It was shown in [3][4][8], that the mathematical modeling corresponding to an one - dimensional superconductor generates two fundamental parameters, Steckly Parameter α which characterizes the installation and Biot Number B linking conductive and convective flows. Explicitly, α and B are given by:

$$\alpha_{dim=1} = \frac{\rho I_c^2}{AhP_e(T_c - T_b)}, \quad B_{dim=1} = \frac{hP_eL_c^2}{AK^*} \tag{19}$$

From our new mathematical modeling, we propose an extension of α and B for the three dimensional case. This extention permits, following the studies desired, to write the term F as:

$$F(u) = B\Big(i^2\tilde{g}(u) - \frac{1}{\alpha}\tilde{q}(u)\Big)$$

$$F(u) = B\tilde{g}(u)\Big(\alpha i^2 - \frac{\tilde{q}(u)}{\tilde{g}(u)}\Big)$$

4.2. Three dimensional case. Let μ be the domain compactness defined by:

$$\mu = \frac{A}{V}$$

where A is the exchange surface and V the domain volume. The Joule Effect G is given by the formula:

$$G(T) = \frac{\rho I^2}{(V\mu)^2} \tag{20}$$

where ρ is the electric resistivity supposed known for a given superconductors. Then we introduce a reduction as follow:

$$g(T) = \widetilde{g}(u) = \frac{(V\mu)^2}{\rho I^2}G(T) \tag{21}$$

Transferred power Q is :

$$Q(T) = h\mu(T_c - T_b) \tag{22}$$

h is the Kapitza coefficient, from (20) we introduce :

$$q(T) = \widetilde{q}(u) = \frac{Q(T)}{h\mu(T_c - T_b)} \tag{23}$$

We have:

$$W(T) = G(T) - Q(T) \quad et \;\; W = W^*F$$

Which gives:

$$W = W^*F(u) = \frac{\rho I^2}{(V\mu)^2}\widetilde{g}(u) - h\mu(T_c - T_b)\widetilde{g}(u) \tag{24}$$

We put $F^*(u) = \lambda F(u)$, if we replace in (24), we obtain:

$$F^*(u) = \frac{L^*}{T^*K^*}\frac{\rho i^2 I_c^2}{(V\mu)^2}\widetilde{g}(u) - \frac{L^*}{T^*K^*}h\mu(T_c - T_b)\widetilde{q}(u) \tag{25}$$

In other words, we were able to put F^* in the form:

$$F^*(u) = B\big(\alpha i^2 g(u) - q(u)\big)$$

With

$$B = \frac{L^*h}{K^*}\mu \quad and \quad \alpha = \frac{\rho I_c^2}{(T_c - T_b)h}\frac{1}{V^2\mu^3} \tag{26}$$

We define the parameter γ by:

$$\gamma = \alpha B = \frac{\rho I_c^2 L^*}{V^2 \mu^2 K^*(T_c - T_b)} \tag{27}$$

In conclusion, by introducing the compactness coefficient and by technics of dimensional analysis, we have obtain three fundamental parameters having a great interest in the study of thermal stability of superconductor.α, B and γ.

3. Conclusion. In this paper, we present a new mathematic modeling of the thermic state of a three dimensional superconductor. By introducing a geometric quantity linking surface and volume, this formulation permits to rewrite, in dimension 3, Steckly parameter and Biot Number.

3. Bibliographie.

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