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Oscillation Criteria for Third-Order Nonlinear Neutral Delay Dynamic Equations on Time Scales

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Abstract

In this paper, we establish some sufficient conditions for oscillation of solutions of third order nonlinear neutral delay dynamic equations of the type

$$\left(a(t)\left((x(t)-p(t)x(\delta(t)))^{\Delta^2}\right)^{\gamma}\right)^{\Delta}+q(t)x^{\gamma}(\tau(t))=0$$

are obtained.

Keywords: *Oscillation, Third order, Neutral dynamic equations.*

1. Introduction

We consider the nonlinear neutral delay dynamic equations of the form

$$\left(a(t)\left((x(t)-p(t)x(\delta(t)))^{\Delta^2}\right)^{\gamma}\right)^{\Delta}+q(t)x^{\gamma}(\tau(t))=0 \tag{1}$$

Throughout this paper we assume the following conditions:

(H) γ is a is a ratio of odd positive integers. $a(t),\ p(t),\ q(t)$ are positive real-valued, rd-continuous functions defined on the time scale interval $[a,b]$ (throughout $a,b\in\mathbb{T}$ with $a<b$). $0\le p(t)\le p<1,\ \tau(t)\le t$ and $\delta(t)\le t,$ $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$ are positive and $\int_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\gamma}}(s)}\Delta s=\infty$ and set $z(t)=x(t)-p(t)x(\delta(t))$.

Now, we state and prove some useful lemmas, which we will use in the proofs of our main results. We begin with the following lemma.

2. Main Results

Lemma 2.1. *Let* $x(t)$ *be a positive solution of (1). Then there are only the following cases for* $z(t)=x(t)-p(t)x(\delta(t))$ *satisfies.*

(i) $z(t)>0, z^{\Delta}(t)>0, z^{\Delta^2}(t)>0,$

(ii) $z(t)>0, z^{\Delta}(t)<0, z^{\Delta^2}(t)>0,$

(iii) $z(t)<0, z^{\Delta}(t)<0, z^{\Delta^2}(t)>0,$

(iv) $z(t)<0, z^{\Delta}(t)<0, z^{\Delta^2}(t)<0, t\in[t_1,\infty)$ *where* t_1 *is sufficiently large.*

Proof: Assume that $x(t)$ is a positive solution of (1) on $t\in[t_0,\infty)$. Then

$$\left(a(t)\left((z(t))^{\Delta^2}\right)^{\gamma}\right)^{\Delta}=-q(t)x^{\gamma}(\tau(t))<0 \tag{2}$$

$a(t)\left((z(t))^{\Delta^2}\right)^{\gamma}$ is decreasing and of one sign. Therefore $z^{\Delta^2}(t)$ is also of one sign. We have two possibilities; $z^{\Delta^2}(t)<0$ or $z^{\Delta^2}(t)>0$ for $t\in[t_1,\infty)$ by (2). If we choose $z^{\Delta^2}(t)<0,$ then there exists a constant $M>0$ such that

$$a(t)\left((z(t))^{\Delta^2}\right)^{\gamma}\le -M<0.$$

Integrating the above inequality from t_1 to t, we obtain

$$z^{\Delta}(t)\le z^{\Delta}(t_1)-M^{\frac{1}{\gamma}}\int_{t_1}^{t}\frac{1}{a^{\frac{1}{\gamma}}(s)}\Delta s.$$

Letting $t \to \infty$ and using (H), we get $z^{\Delta}(t) \to -\infty$. But $z^{\Delta^2}(t) < 0$ and $z^{\Delta}(t) < 0$ eventually, imply $z(t) < 0$. Thus for $z^{\Delta^2}(t) < 0$ case (iv) may occur.

On the other hand if $z^{\Delta^2}(t) > 0$, then $z^{\Delta}(t) < 0$ is of one sign. If $z^{\Delta}(t) > 0$, $t \ge t_1$, then $z(t) > 0$. So for $z^{\Delta^2}(t) > 0$ only for the cases (i),(ii),(iii) may occur. The proof is complete.

Lemma 2.2. *Let $x(t)$ be a positive solution of equation (1) and the corresponding $z(t)$ satisfy case (ii) of Lemma 2.1. If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta u \Delta v = \infty, \qquad (3)$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$

Proof. Suppose that $x(t)$ is a positive solution of (1). It is clear that there exists a finite limit, $\lim_{t\to\infty} z(t) = \ell$. We shall prove that $\ell = 0$. Assume that $\ell > 0$. It follows that $z(t) < x(t)$. Combing this fact with (1), we are led to

$$\left(a(t)\left(\left(z(t)\right)^{\Delta^2}\right)^{\gamma}\right)^{\Delta} \le -q(t) z^{\gamma}(\tau(t)) \le -\ell^{\gamma} q(t).$$

We get

$$z^{\Delta^2}(t) \ge \ell\left(\frac{1}{a(t)}\int_{t}^{\infty} q(s)\Delta s\right)^{\frac{1}{\gamma}}.$$

Integrating again from t to ∞, we have

$$-z^{\Delta}(t) \ge \ell \int_{t}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta u.$$

Again taking Integrating t_1 to ∞, we obtain

$$z(t_1) \ge \ell \int_{t_1}^{\infty}\int_{u}^{\infty}\left(\frac{1}{a(u)}\int_{v}^{\infty} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta u \Delta v.$$

This contradicts (3). Then $\ell = 0$. Moreover the boundedness of $x(t)$ yields $\limsup_{t\to\infty} x(t) = a$; $0 \le a < \infty$. Then there exists a sequence t_k such that $\lim_{t\to\infty} t_k = \infty, \lim_{t\to\infty} x(t_k) = a$.

If $a > 0$, choosing $\varepsilon = \frac{a(1-p)}{2p}$ we see that $x(\delta(t_k)) < a + \varepsilon$ eventually. Moreover

$$0 = \lim_{k\to\infty} z(t_k) \ge \lim_{k\to\infty}(x(t_k) - p(a+\varepsilon))$$
$$= \frac{a}{2}(1-p) > 0.$$

Thus $a = 0$ and that $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

Lemma 2.3. *[5] Assume that $u(t) > 0,\ u^{\Delta}(t) \ge 0,\ u^{\Delta^2}(t) \le 0$ for all $t \in [t_0, \infty)$. Then for each $\ell \in (0,1)$ there exists an integer $T \ge t_0$ such that $\frac{u(\tau(t))}{\tau(t)} \ge \ell \frac{u(t)}{t}$ for $t \ge T$.*

Lemma 2.4. *[5] Assume that $z(t) > 0, z^{\Delta}(t) > 0, z^{\Delta^2}(t) > 0, z^{\Delta^3}(t) \le 0$ for all $t \ge T$. Then $\frac{z(t)}{z^{\Delta}(t)} \ge \frac{t-T}{2}$ for $t \ge T$.*

Lemma 2.5. *[5] Assume that $z^{\Delta}(t) > 0,\ z^{\Delta^2}(t) > 0,\ z^{\Delta^3}(t) \le 0$ for all $t \ge T$. Then*

$$(t-T)\frac{z^{\Delta^2}(t)}{z^{\Delta}(t)} \le 1 \quad \textit{for} \quad t \ge T.$$

Now, we present the oscillation results. For simplicity, we introduce the following notation

$$\tilde{p}_* = \liminf_{t\to\infty} \frac{t^{\gamma}}{a(t)}\int_{t}^{\infty} \tilde{P}_{\ell}(s)\Delta s,$$
$$\tilde{q}_* = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \frac{s^{\gamma+1}}{a(s)} \tilde{P}_{\ell}(s)\Delta s, \qquad (4)$$

Where $\tilde{P}_{\ell}(s) = \ell^{\gamma} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\left(\frac{\tau(s)-T}{2}\right)^{\gamma}$ with $\ell \in (0,1)$ arbitrarily chosen and T large enough. Moreover for $z(t)$ satisfying case (i), we define

$$w(t) = a(t)\left(\frac{z^{\Delta^2}(t)}{z^{\Delta}(t)}\right)^{\gamma} \qquad (5)$$

and

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$$r=\liminf_{t\to\infty}\frac{t^{\gamma}w^{\sigma}(t)}{a^{\sigma}(t)} \text{ and } R=\limsup_{t\to\infty}\frac{t^{\gamma}w(t)}{a(t)}. \qquad (6)$$

Lemma 2.6. *Assume that $a(t)$ is non decreasing. Let $x(t)$ be a positive solution of equation (1)*

(I) Let $\tilde{p}_* < \infty$ and $\tilde{q}_* < \infty$. Suppose that the corresponding $z(t)$ satisfies case (i) of Lemma 2.1. Then

$$\tilde{p}_* \le r - r^{1+\frac{1}{\gamma}} \text{ and } \tilde{p}_* + \tilde{q}_* \le 1. \qquad (7)$$

(II) If $\tilde{p}_* = \infty$ or $\tilde{q}_* = \infty$, then $z(t)$ does not belong under case (i) of Lemma 2.1.

Proof. Part (I): Assume that $x(t)$ is a positive solution of equation (1) and the corresponding $z(t)$ satisfies (i). Since that $0 < z(t) < x(t)$, we obtain

$$\left(a(t)((z(t))^{\Delta^2})^{\gamma}\right)^{\Delta} < -q(t)z^{\gamma}(\tau(t)) < 0. \qquad (8)$$

Since $a^{\Delta}(t) \ge 0$ we have $z^{\Delta^3}(t) \le 0$. So there exists a $T \ge t_0$ such that $z(t)$ satisfies $z(\tau(t)) > 0,\ z^{\Delta}(t) > 0,\ z^{\Delta^2} > 0,\ z^{\Delta^3}(t) \le 0$, for $t \ge T$. From the definition of $w(t)$ and (8) we see that $w(t) > 0$ and satisfies,

$$w^{\Delta}(t) \le \frac{-q(t)z^{\gamma}(\tau(t))}{(z^{\Delta}(t))^{\gamma}} - \frac{\gamma}{a^{\frac{1}{\gamma}\sigma}(t)} w^{\frac{\gamma+1}{\gamma}\sigma}(t) \qquad (9)$$

From Lemma 2.3 with $u(t) = z^{\Delta}(t)$, we have for ℓ the same as in $\tilde{P}_{\ell}$

$$\frac{1}{z^{\Delta}(t)} \ge \frac{\ell\tau(t)}{t}\frac{1}{z^{\Delta}(\tau(t))},\ t \ge T,$$

which with (9) gives

$$w^{\Delta}(t) \le -\ell^{\gamma}q(t)\left(\frac{\tau(t)}{t}\right)^{\gamma}\left(\frac{z(\tau(t))}{z^{\Delta}(\tau(t))}\right)^{\gamma} - \frac{\gamma}{a^{\frac{1}{\gamma}\sigma}(t)} w^{\frac{\gamma+1}{\gamma}\sigma}(t)$$

Using the fact form Lemma 2.4 that $z(t) \ge \frac{(t-T)}{2} z^{\Delta}(t)$ we have,

$$w^{\Delta}(t) + \tilde{P}_{\ell}(t) + \frac{\gamma}{a^{\frac{1}{\gamma}\sigma}(t)} w^{\frac{\gamma+1}{\gamma}\sigma}(t) \le 0. \qquad (10)$$

Since $\tilde{P}_{\ell}(t) > 0$ and $w(t) > 0$ for $t \ge T$, $w^{\Delta}(t) \le 0$ and

$$\frac{-w^{\Delta}(t)}{\gamma w^{\frac{\gamma+1}{\gamma}\sigma}(t)} \ge \frac{1}{a^{\frac{1}{\gamma}\sigma}(t)} \text{ for } t \ge T.$$

This implies that

$$\left(\frac{1}{w^{\frac{1}{\gamma}}(t)}\right)^{\Delta} \ge \frac{1}{a^{\frac{1}{\gamma}\sigma}(t)}$$

Integrating the last inequality T to t and using the fact that $w(t)$ is decreasing, we obtain

$$w(t) < \frac{1}{\left(\gamma\int_T^t \frac{1}{a^{\frac{1}{\gamma}\sigma}(s)}\Delta s\right)^{\gamma}} \qquad (11)$$

which view of (H) implies that, $\lim_{t\to\infty} w(t) = 0.$ On the other hand, from the definition of $w(t)$ and Lemma 2.5, we see that

$$0 \le r \le R \le 1. \qquad (12)$$

Now, we prove that the first inequality in (7) holds. Let $\varepsilon > 0$, then from the definition of $\tilde{p}_*$ and r, we can pick up $t_2 \in [T,\infty)$ sufficiently large $\frac{t^{\gamma}}{a(t)}\int_t^{\infty}\tilde{P}_{\ell}(s)\Delta s \ge \tilde{p}_* - \varepsilon$ and $\frac{t^{\gamma}w^{\sigma}(t)}{a^{\sigma}(t)} \ge r - \varepsilon$ for $t \in [t_2,\infty)$. Integrating (10) from t to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \ge \int_t^{\infty}\tilde{P}_{\ell}(s)\Delta s + \gamma\int_t^{\infty}\frac{w^{\frac{\gamma+1}{\gamma}\sigma}(s)}{a^{\frac{1}{\gamma}\sigma}(s)}\Delta s,\ \ for\ \ t \ge t_2$$

Using the fact $a^{\Delta}(t) \ge 0,$ it follows from (13) that

$$\frac{t^{\gamma}w(t)}{a(t)} \ge (\tilde{p}_* - \varepsilon) + \frac{\gamma t^{\gamma}}{a(t)}\int_t^{\infty}\frac{s^{\gamma+1}a^{\sigma}(s)w^{\frac{\gamma+1}{\gamma}\sigma}(s)}{s^{\gamma+1}a^{\frac{\gamma+1}{\gamma}\sigma}(s)}\Delta s$$

and so

$$\frac{t^\gamma w(t)}{a(t)} \geq (\tilde{p}_* - \varepsilon) + t^\gamma (r-\varepsilon)^{1+\frac{1}{\gamma}} \int_t^\infty \frac{\gamma}{s^{\gamma+1}} \Delta s. \qquad (14)$$

From (14), we have

$$\frac{t^\gamma w(t)}{a(t)} \geq (\tilde{p}_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\gamma}}$$

Taking lim inf of both sides as $t \to \infty$, we get

$$r \geq (\tilde{p}_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\gamma}}$$

Since, $\varepsilon > 0$ is arbitrary we get the result,

$$\tilde{p}_* \leq r - r^{1+\frac{1}{\gamma}} \qquad (15)$$

To complete the proof of part (I), it remains to prove second inequality in (7). Multiplying the inequality (10) by $\frac{t^{\gamma+1}}{a(t)}$ and Integrating from t_2 to t, we obtain

$$\int_{t_2}^t \frac{s^{\gamma+1} w^\Delta(s)}{a(s)} \Delta s - \int_{t_2}^t \frac{s^{\gamma+1}}{a(s)} \tilde{P}_\ell(s) \Delta s - \gamma \int_{t_2}^t \left(\frac{s^\gamma w^\sigma(s)}{a^\sigma(s)} \right)^{\frac{\gamma+1}{\gamma}} \Delta s. \qquad (16)$$

By integration by parts, we obtain

$$\frac{t^{\gamma+1}}{a(t)} w(t) \leq \frac{t_2^{\gamma+1}}{a(t_2)} w(t_2) - \int_{t_2}^t \frac{s^{\gamma+1}}{a(s)} \tilde{P}_\ell(s) \Delta s - \gamma \int_{t_2}^t \left(\frac{s^\gamma w^\sigma(s)}{a^\sigma(s)} \right)^{\frac{\gamma+1}{\gamma}} \Delta s + \int_{t_2}^t w^\sigma(s) \left(\frac{s^{\gamma+1}}{a(s)} \right)^\Delta \Delta s.$$

Since $a^\Delta(t) \geq 0$, we have

$$\left(\frac{s^{\gamma+1}}{a(s)} \right)^\Delta = \frac{a(s)(s^{\gamma+1})^\Delta - s^{\gamma+1} a^\Delta(s)}{a(s) a^\sigma(s)} \leq \frac{(\gamma+1)\sigma(s)^\gamma}{a^\sigma(s)}$$

Hence,

$$\frac{t^{\gamma+1} w(t)}{a(t)} \leq \frac{t_2^{\gamma+1}}{a(t_2)} w(t_2) - \int_{t_2}^t \frac{s^{\gamma+1} \tilde{P}_\ell(s)}{a(s)} \Delta s + \int_{t_2}^t \left[\frac{(\gamma+1)\sigma(s)^\gamma w^\sigma(s)}{a^\sigma(s)} - \gamma \left(\frac{s^\gamma w^\sigma(s)}{a^\sigma(s)} \right)^{\frac{\gamma+1}{\gamma}} \right] \Delta s.$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$$

with $u = \frac{s^\gamma w^\sigma(s)}{a^\sigma(s)} > 0, \; A = \gamma$ and $B = (\gamma+1)\left(\frac{\sigma(s)}{s} \right)^\gamma$, we get

$$\frac{t^{\gamma+1}}{a(t)} w(t) \leq \frac{t_2^{\gamma+1}}{a(t_2)} w(t_2) - \int_{t_2}^t \frac{s^{\gamma+1} \tilde{P}_\ell(s)}{a(s)} \Delta s + \int_{t_2}^t \left(\frac{\sigma(s)}{s} \right)^{\gamma(\gamma+1)} \Delta s$$

It follows that

$$\frac{t^\gamma w(t)}{a(t)} \leq \frac{t_2^{\gamma+1} w(t_2)}{t a(t_2)} - \frac{1}{t} \int_{t_2}^t \frac{s^{\gamma+1} \tilde{P}_\ell(s)}{a(s)} \Delta s + \frac{1}{t} \int_{t_2}^t \left(\frac{\sigma(s)}{s} \right)^{\gamma(\gamma+1)} \Delta s. \qquad (17)$$

Taking the lim sup of $t \to \infty$

$$R \leq -\tilde{q}_* + 1.$$

Combining this with (12), we have

$$\tilde{p}_* \leq r - r^{1+\frac{1}{\gamma}} \leq r \leq R \leq -\tilde{q}_* + 1$$

which gives the desired second inequality in (7). The proof of part (I) is complete.

Part (II): Assume $x(t)$ positive solution of (1) and that the corresponding function $z(t)$ satisfies (i). First assume $\tilde{p}_* = \infty$. This is exactly as in the proof of the first part, we obtain (10). Then

$$\frac{t^\gamma}{a(t)}w(t) \ge \frac{t^\gamma}{a(t)}\int_t^\infty \tilde{P}_\ell(s)\Delta s.$$

Taking lim inf of both sides at $t \to \infty$, we obtain in view of (14)

$$1 \ge r \ge \infty.$$

This is a contradiction. Next we assume that $\tilde{q}_* = \infty$. Then taking lim inf and lim sup on the left and right sides of (17) respectively, we obtain

$$0 \le R \le -\infty .$$

This contradiction completes the proof.
Now we are ready to present the following oscillation criterion for equation (1).

Theorem 2.7. *Assume that condition (3) holds and* $a(t)$ *is nondecreasing. Let* $x(t)$ *be a solution of (1). If*

$$\tilde{p}_* = \liminf_{t\to\infty} \frac{t^\gamma}{a(t)}\int_t^\infty \tilde{P}_\ell(s)\Delta s > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \qquad (18)$$

Then $x(t)$ *is oscillatory or* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1). Without loss of generality we may assume that $x(t)$ is a positive solution of equation (1). We claim that $x(t)$ is bounded. To prove this we assume, on the contrary, that $x(t)$ is unbounded. Hence there exists sequence t_m such that $\lim_{m\to\infty} t_m = \infty$; moreover $\lim_{m\to\infty} x(t_m) = \infty$ and

$$x(t_m) = \max\{x(s); t_0 \le s \le t_m\}.$$

Since $\delta(t) \to \infty$ *as* $t \to \infty$, we can choose m sufficiently large that $\delta(t_m) > t_0$. As that $\delta(t) \le t$, we have

$$x(\delta(t_m)) \le \max\{x(s); t_0 \le s \le \delta(t_m)\}$$
$$\le \max\{x(s); t_0 \le s \le t_m\} = x(t_m).$$

Therefore for all large m

$$z(t_m) = x(t_m) - p(t_m)x(\delta(t_m)) \ge (1-p)x(t_m).$$

Thus $z(t_m) \to \infty$ *as* $m \to \infty$. So $z(t)$ is positive and unbounded. It follows from Lemma2.1 of (i) has to hold Lemma 2.6 (I) provides

$$\tilde{p}_* \le r - r^{1+1/\gamma}.$$

Using the inequality $Bu - Au^{1+1/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}B^{\gamma+1}/A^\gamma$ with A = B = 1 and $u = r$, we obtain that

$$\tilde{p}_* \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}.$$

Which contradicts (18). So we conclude that both $x(t)$ and $z(t)$ are bounded. Lemma 2.1 now implies that for $z(t)$ either (ii) or (iii) holds.
If the case (ii) holds then Lemma 2.2 ensures that $\lim_{t\to\infty} x(t) = 0$. On the other hand if the case (iii) holds then there exists a finite $\lim_{t\to\infty} z(t) = -c < 0$. We know that $0 < x(t)$ is bounded, so

$$\limsup_{t\to\infty} x(t) = a, \qquad 0 \le a < \infty .$$

We claim that $a = 0$. If not then there exists a sequence t_k such that $\lim_{k\to\infty} x(t) = \infty$, *and* $\lim_{k\to\infty} x(t_k) = a$. It is easy to see that for $\varepsilon = \frac{a(1-p)}{2p} > 0$, we have $x(\delta(t_k)) < a + \varepsilon$. Moreover

$$0 > -c = \lim_{k\to\infty} z(t_k) \ge \lim_{k\to\infty}(x(t_k) - p(a+\varepsilon))$$
$$= \frac{a}{2}(1-p) > 0.$$

This is a contradiction. Thus $a = 0$ and $\lim_{t\to\infty} x(t) = 0$.
This completes the proof.

Corollary 2.8. *Assume that condition (3) holds and* $a(t)$ *is non decreasing. Let* $x(t)$ *be a solution of (1). If*

$$\liminf_{t\to\infty} \frac{t^\gamma}{a(t)}\int_t^\infty q(s)\frac{\tau^{2\gamma}(s)}{s^\gamma}\Delta s > \frac{(2\gamma)^\gamma}{(\gamma+1)^{\gamma+1}} \qquad (19)$$

then $x(t)$ *is oscillatory or* $x(t) \to 0$ *as* $t \to \infty$.

Proof. We shall show that condition (19) implies condition (18). First note that for any $\ell \in (0,1)$ there exists an integer t_1 such that $\tau(t) - T \ge \ell\tau(t),\ t \ge t_1$. Therefore,

$$\tilde{P}_\ell(t) \ge \frac{\ell^{2\gamma}}{2^\gamma}\frac{\tau^{2\gamma}(t)}{t^\gamma}q(t),\ t \ge t_1. \qquad (20)$$

On the other hand, (19) implies that for some $\ell \in (0,1)$

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{a(t)} \int_{t}^{\infty} q(s) \frac{\tau^{2\gamma}(s)}{s^{\gamma}} \Delta s > \frac{1}{\ell^{2\gamma}} \frac{(2\gamma)^{\gamma}}{(\gamma+1)^{\gamma+1}}, \quad (21)$$

Combining (20) with (21), we obtain (18).

Theorem 2.9. *Assume that condition (3) holds and* $a(t)$ *is non-decreasing. Let* $x(t)$ *be a solution of equation (1). If*

$$\tilde{p}_* + \tilde{q}_* > 1, \quad (22)$$

then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

As a consequence of Theorem 2.9, we have the following results.

Corollary 2.10*. Assume that condition (3) holds and* $a(t)$ *is non decreasing. Let* $x(t)$ *be a solution of equation (1). If*

$$\tilde{q}_* = \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\gamma+1}}{a(s)} \tilde{P}_{\ell}(s) \Delta s > 1, \quad (23)$$

Then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0.$

As a matter of fact we can again slightly simplify function $\tilde{p}_{\ell}(t)$ in (23).

Corollary 2.11. *Assume that condition (3) holds and* $a(t)$ *is non-decreasing. Let* $x(t)$ *be a solution of equation (1). If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s\tau^{2\gamma}(s)}{a(s)} q(s) \Delta s \geq 2^{\gamma} \quad (24)$$

Then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0.$

The proof is similar to that of Corollary 2.8 and hence the details are omitted.

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