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On Identities of Rogers-Ramanujan Type

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Abstract: The most famous of the "Series = Product" Identities are

For |q|<1, $$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 2 \,(\text{mod } 5) = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 1 (\text{mod } 5) = \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}$$

where $(q;q)_n=\prod_{j=1}^{n}(1-q^j)$, $(q;q)_\infty=\prod_{j=1}^{\infty}(1-q^j)$

and$(a_1,a_2,\ldots.a_s;q)_\infty= (a_1;q)_\infty\,(a_2;q)_\infty\ldots..\,(a_s;q)_\infty$

which are known as the celebrated original Rogers-Ramanujan Identity. These two identities have motivated extensive research over the past hundred years. They were first proved by L. J. Rogers in 1894 that was completely ignored. They were rediscovered without proof by Ramanujan sometime before 1913. Also in 1917, these identities were rediscovered and proved independently by Issai Schur. There are now many different proofs of these Identities. In the ensuing decades, numerous identities that are similar to the Rogers-Ramanujan Identities has been discovered by several eminent mathematicians like Jackson, W. N. Bailey, G. E. Andrews, L. J. Slater, A.K. Agarwal, etc.

The Rogers-Ramanujan Identities have two aspects: one analytical and the other is combinatorial. In this present paper, some identities of Rogers-Ramanujan Type related to modulo 6,7 and10 has been derived by using some general transformation between Basic Hypergeometric Series and with the incorporation of some identities from Lucy Slater's famous list of 130 identities of Rogers-Ramanujan type.

Key words: Rogers-Ramanujan Identity, Slater's Identity, Basic Hypergeometric Series, Jacobi's Triple Product Identity, Bailey Pair etc.

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1. Introduction:

For |q|<1, the q-shifted factorial is defined by

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ \text{for } n\geq 1$$

$$\text{and}\quad (a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k).$$

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,....,a_m;q)_n = (a_1;q)_n(a_2;q)_n....(a_m;q)_n$$

$$(a_1,a_2,....,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty....(a_m;q)_\infty.$$

The Basic Hyper geometric Series is

$${}_{p+1}\phi_{p+r}\left(\begin{matrix} a_1,a_2,.......,a_{p+1};q;x \\ b_1,b_2,.......,b_{p+r}\end{matrix}\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n....(a_{p+1};q)_n x^n(-1)^{nr}q^{\frac{n(n-1)r}{2}}}{(q;q)_n(b_1:q)_n(b_2;q)_n.....(b_{p+r};q)_n}$$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when |x|<1.

<u>Some definitions:</u>

<u>Ramanujan's Theta function</u>: Ramanujan's Theta function ([4], P.11, Eq. (1.1.5)) is defined as

$$f\,(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},\ \text{for } |ab|<1.$$

The following special cases of f (a, b) arise so often that they were given their own notation by Ramanujan ([4], P.11):

$$\varphi(q) = f\,(q,q)$$

$$\psi(q) = f\,(q,q^3)$$

$$f\,(-q) = f\,(-q,-q^2)$$

Jacobi's triple product identity :(See [7], P.2, Eq. (1.1.7))

For |ab| < 1, f (a, b) $=(-a,-b,ab;ab)_\infty$,

where $(a;w)_n = \prod_{n=0}^{\infty}(1-aw^n)$, and $(a_1,a_2,\ldots.a_r;w)_\infty=(a_1;w)_\infty(a_2;w)_\infty\ldots..(a_r;w)_\infty$

An immediate corollary ([7], P-2, Eq. (1.1.8), (1.1.9), (1.1.10)) of this identity is thus

$$f(-q) = (q;q)_\infty$$

$$\varphi(q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

Sometimes a linear combination of two theta series can be expressed as a single infinite product as follows: (See [7], p-2, Eq. (1.1.12))

$$(Q,x) = f(-wx^3,-w^2x^{-3}) + xf(-wx^{-3},-w^2x^3)$$

$$= \frac{f(wx^{-1},x)\,f(-wx^{-2},-wx^2)}{f(-w^2,-w^4)}$$

$$= (-wx^{-1},-x,w;w)_\infty(wx^{-2},wx^2;w^2)_\infty$$

2. We now list some general transformations. These can be derived as limiting case of transformations between basic hyper geometric series. Let a, b, c, d, γ and q ∈**C**, |q|<1. Then

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n q^{n(n-1)/2}(-c\gamma/ab)^n}{(c,\gamma,q;q)_n} = \frac{(c\gamma/ab;q)_\infty}{(\gamma;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/a,c/b;q)_n q^{n(n-1)/2}(-\gamma)^n}{(c,c\gamma/ab,q;q)_n} \qquad (2.1)$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(-\gamma)^n}{(b;q)_n(q;q)_n} = (\gamma;q)_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2-3n)/2}(-b\gamma)^n}{(q;q)_n\,(b;q)_n(\gamma;q)_n} \qquad (2.2)$$

$$(-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{q^{(m^2+m)/2} a^m}{(-bq^n;q^n)_m (q;q)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{q^{n(m^2+m)/2} (b)^m}{(-aq;q)_{nm} (q^n;q^n)_m} \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n^2-n} (-b)^n}{(q;q)_n (ab;q^2)_n} = \frac{(b;q^2)_n}{(ab;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n q^{n^2-n} (-bq)^n}{(q^2;q^2)_n (b;q^2)_n} \tag{2.4}$$

The transformation (2.1) is a limiting case of a q-analogue of the Kummer Thomae-Whipple formula (see [5] P-72, Equation 3.2.7) or (see [7] p-40, Eq. (6.1.2)). The proof of transformation (2.2) is found in [6]. This transformation (2.2) is also appears in [7], (Equation (6.1.11) p-41).A limiting case of a transformation due to Andrews [2] leads to the identity (2.3). This transformation (2.3) is also appears in [7], (Equation (6.1.14) p-41). The identity (2.4) follows from a result of Andrews in [3]. It is also appears as Equation (6.1.19) in [7].

Now we introduce some identities from the Lucy Slater's famous list [9] of Rogers-Ramanujan Type Identities. Each of them below that appears in [9] is designated with a "Slater number" S.n.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)} (-q)_n}{(q;q^2)_{n+1} (q)_n} = \frac{f(-q,-q^5)}{\varphi(-q)}, \text{ (see [7], Equation (2.6.5), p-13); } (s._{22}) \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n(n+1)/2}}{(q;q^2)_{n+1} (q)_n} = \frac{f(-q^2,-q^5)}{f(-q)}, \text{ (see [7], Equation (2.10.4), p-17); } (s._{44}) \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q^2)_{n+1} (q)_n} = \frac{f(-q^4,-q^6)}{f(-q)}, \text{ (see [7], Equation (2.10.5), p-17); } (s._{46}) \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (-q)_n}{(q)_{2n+1}} = \frac{Q(q^7,-q^2)}{\varphi(-q)}, \text{ (see [7], Equation (2.14.5), p-19); } (s._{80}) \tag{2.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (-q)_n}{(q)_{2n}} = \frac{Q(q^7,-q)}{\varphi(-q)}, \text{ (see [7], Equation (2.14.4), p-19); } (s._{81}) \tag{2.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)/2} (-q)_n}{(q)_{2n+1}} = \frac{Q(q^7,-q^3)}{\varphi(-q)}, \text{ (see [7], Equation (2.14.6), p-19); } (s._{82}) \tag{2.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n+1}} = \frac{Q(q^{10},-q^3)}{f(-q)}, \text{ (see [7], Equation (2.20.5), p-23); } (s._{94}) \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q)_{2n+1}} = \frac{Q(q^{10},-q^4)}{f(-q)}, \text{ (see [7], Equation (2.20.6), p-23); } (s._{96}) \tag{2.8}$$

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$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n}} = \frac{Q(q^{10},-q)}{f(-q)}$$, (see [7], Equation (2.20.3), p-22); ($S._{99}$) (2.9)

3. Some Identities of Rogers-Ramanujan Type related to modulo 10:

Replacing q by q^2 in (2.2) we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(b;q^2)_n(q^2;q^2)_n} = (\gamma;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2-3n)}(-b\gamma)^n}{(q^2;q^2)_n(b;q^2)_n(\gamma;q^2)_n} \quad (3.1)$$

Setting $b = q^3, \gamma = -q^3$ in (3.1) we get,

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^3;q^2)_n(q^2;q^2)_n} = (1-q^2)(-q^3;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)}}{(q^2;q^4)_{n+1}(q^2;q^2)_n}$$

which for $q \to q^{1/2}$ gives:

$$\frac{1}{(1-q)(-q^{3/2};q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+2)/2}}{(q^{3/2};q)_n(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)/2}}{(q;q^2)_{n+1}(q;q)_n}$$

$$= \frac{f(-q^2,-q^8)}{f(-q)}, \text{ (on using (2.2))}$$

$$= \frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q;q)_\infty}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \ n \not\equiv 0, 2, 8 \ (\text{mod } 10) \quad (3.2)$$

The equation (3.1) for $b = q, \gamma = -q$ gives

$$\frac{1}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{(3n^2-n)}}{(q^2;q^2)_n(q^2;q^4)_n}$$

which for $q \to q^{1/2}$ gives:

$$\frac{1}{(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q^{1/2};q)_n(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{(3n^2-n)/2}}{(q;q^2)_n(q;q)_n}$$

$$=\frac{f(-q^4,-q^6)}{f(-q)}, \text{ (on using (2.3))}$$

$$=\frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q;q)_\infty}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, \text{ n} \not\equiv 0, 4, 6 \text{ (mod 10)} \qquad (3.3)$$

Also the equation (3.1) for $b = q^3, \gamma = -q^2$ gives

$$\frac{(-q^2;q^2)_\infty}{(1-q^2)}\sum_{n=0}^{\infty}\frac{q^{3n^2+2n}}{(q^4;q^4)_n(q^3;q^2)_n} = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_{2n+1}}$$

$$=\frac{Q(q^{10},-q^3)}{f(-q)}, \text{ (on using (2.7))}$$

$$=\frac{(q^7,q^3,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}$$

Thus we have

$$\frac{(-q^2;q^2)_\infty}{(q^4,q^{16};q^{20})_\infty(1-q^2)}\sum_{n=0}^{\infty}\frac{q^{3n^2+2n}}{(q^4;q^4)_n(q^3;q^2)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, \text{ n} \not\equiv 0, 3, 7 \text{ (mod 10)} \qquad (3.4)$$

Moreover the equation (3.1) for $b = q^3, \gamma = -q^3$ gives

$$\frac{(-q^3;q^2)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{3n^2+3n}}{(q^2;q^2)_n(q^6;q^4)_n} = \sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q)_{2n+1}}$$

$$=\frac{Q(q^{10},-q^4)}{f(-q)}, \text{ (on using (2.8))}$$

$$=\frac{(q^6,q^4,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty}$$

So, we have

$$\frac{(-q^3;q^2)_\infty}{(1-q)(q^2,q^{18};q^{20})_\infty}\sum_{n=0}^{\infty}\frac{q^{3n^2+3n}}{(q^2;q^2)_n(q^6;q^4)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, \text{ n} \not\equiv 0, 4, 6 \text{ (mod 10)} \qquad (3.5)$$

And, finally setting $b = q^3$,$\gamma = -q^2$ in (3.1), it yields

$$(-q^2, q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n (q^4;q^4)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_n (q^2;q^2)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}}$$

$$= \frac{Q(q^{10}, -q)}{f(-q)} \text{ (on using (2.9))}$$

$$= \frac{(q^9, q, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q;q)_\infty}$$

Thus we have the following identity

$$\frac{(-q^2;q^2)_\infty}{(q^8,q^{12};q^{20})_\infty} \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n (q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \ n \not\equiv 0, 1, 9 \ (\text{mod } 10) \qquad (3.6)$$

4. *Some Identities of Rogers-Ramanujan Type related to modulo 7:*

Again, replacing q by q^2 in (2.3) we get

$$(-bq^{2n}; q^{2n})_\infty \sum_{m=0}^{\infty} \frac{q^{m^2+m} a^m}{(-bq^{2n};q^{2n})_m (q^2;q^2)_m} = (-aq^2; q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{n(m^2+m)} (b)^m}{(-aq^2;q^2)_{nm} (q^{2n};q^{2n})_m} \qquad (4.1)$$

The equation (4.1) for $n = 2, a = q^2$ and $b = -q^2$ gives

$$(q^6; q^4)_\infty \sum_{m=0}^{\infty} \frac{q^{m^2+3m}}{(q^6;q^4)_m (q^2;q^2)_m} = (q^4; q^2)_\infty \sum_{m=0}^{\infty} \frac{(-1)^m q^{2(m^2+2m)}}{(-q^4;q^2)_{2m} (q^4;q^4)_m}$$

This, after some simplification gives

$$\frac{(-q^4;q^2)_\infty}{(1-q^2)(q^6;q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2(m^2+2m)}}{(-q^4;q^2)_{2m} (q^4;q^4)_m} = \sum_{m=0}^{\infty} \frac{q^{m^2+3m} (-q^2;q^2)_m}{(q^2;q^2)_{2m+1}}$$

Now taking $q \to q^{1/2}$ and writing n in place of m, we obtain the following identity:

$$\frac{(-q^2;q)_\infty}{(1-q)(q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2+2n)}}{(-q^2;q)_{2n} (q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2} (-q;q)_n}{(q;q)_{2n+1}}$$

$$=\frac{Q(q^7,-q^3)}{\varphi(-q)} \text{, (on using (2.6))}$$

$$=\frac{(q^4,q^3,q^7;q^7)_\infty (q,q^{13};q^{14})_\infty (-q;q)_\infty}{(q;q)_\infty}$$

$$=(q,q^{13};q^{14})_\infty (-q;q)_\infty \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

$$\text{where } n \not\equiv 0, 3,4 \pmod 7 \qquad (4.2)$$

Again, setting $n = 2, a = 1$and $\mathrm{b} = -{}^{1}/_{\mathrm{q}^2}$ in (4.1) we get on some simplification, the following equation:

$$\frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2}}{(-q^2;q^2)_{2m} (q^4;q^4)_m} = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}(-q^2;q^2)_m}{(q^2;q^2)_{2m}}$$

This equation for $q \to q^{1/2}$ gives the following identity:

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n} (q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q)_{2n}}$$

$$=\frac{Q(q^7,-q)}{\varphi(-q)} \text{, (on using (2.5))}$$

$$=\frac{(q^6,q,q^7;q^7)_\infty (q^5,q^9;q^{14})_\infty (-q;q)_\infty}{(q;q)_\infty}$$

$$=(q^5,q^9;q^{14})_\infty (-q;q)_\infty \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

$$\text{where } n \not\equiv 0, 1,6 \pmod 7 \qquad (4.3)$$

Moreover, the equation (4.1) for $n = 2, a = 1$,$\mathrm{b} = -\mathrm{q}^2$ gives

$$\frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2(m^2+2m)}}{(-q^2;q^2)_{2m} (q^4;q^4)_m} = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}(-q^2;q^2)_m}{(q^2;q^2)_{2m+1}} \qquad (4.4)$$

Taking $q \to q^{1/2}$ in (4.4), it yields

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+2n)}}{(-q;q)_{2n}\,(q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q)_n}{(q;q)_{2n+1}}$$

$$=\frac{Q(q^7,-q^2)}{\varphi(-q)},\text{ (on using (2.4))}$$

$$=\frac{(q^5,q^2,q^7;q^7)_\infty(q^3,q^{11};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}$$

$$==(q^3,q^{11};q^{14})_\infty(-q;q)_\infty\prod_{n=1}^{\infty}\frac{1}{1-q^n},$$

where n≢ 0, 2,5 (mod 7) (4.5)

Lastly, taking $q\rightarrow q^{1/2}$ in the transformation (2.4) and then setting

$a=-q^{1/2}$,b = −q, it gives

$$\frac{(-q;q)_\infty}{(q^{\frac{1}{2}};q)_\infty}\sum_{n=0}^{\infty}\frac{(-q^{\frac{1}{2}};q)_n q^{(n^2+2n)/2}}{(q;q)_n(-q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{(n^2+n)/2}(-q^{\frac{1}{2}};q^{1/2})_n}{(q^{1/2};q)_{n+1}(q^{\frac{1}{2}};q^{1/2})_n}$$

Now using the identity (2.1) after replacement of q by q^2, it gives the following identity:

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n^2+2n)}}{(q^4;q^4)_n}=\sum_{n=0}^{\infty}\frac{q^{(n^2+n)}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_n}$$

$$=\frac{f(-q,-q^5)}{\varphi(-q)}$$

$$=\frac{(q,q^5,q^6;q^6)_\infty(-q;q)_\infty}{(q;q)_\infty}$$

That is, $\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n^2+2n)}}{(q^4;q^4)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$, n≢ 0, 1, 5(mod 6) (4.6)

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