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Coloring Phenomena of Hamiltonian Graphs

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Abstract:

This paper studies the Hamiltonian coloring and Hamiltonian chromatic number for different graphs .the main results are1.For any integer n greater than or equal to three, Hamiltonian chromatic number of C_n is equal to n-2. 2. G is a graph obtained by adding a pendant edge to Hamiltonian graph H, and then Hamiltonian chromatic number of G is equal to n-1. 3. For every connected graph G of order n greater than or equal to 2, Hamiltonian chromatic number of G is not more than one increment of square of (n-2). **Mathematics Subject Classification 2000**: 03Exx, 03E10, 05CXX, 05C15, 05C45.

Key words: Chromatic number, Hamiltonian coloring, Hamiltonian chromatic number, pendant edge, spanning connected graph.

§1 Introduction:

Generally in a (d-1) radio coloring of a connected graph G of diameter d, the colors assigned to adjacent vertices must differ by at least d-1, the colors assigned to two vertices whose distance is 2 must differ by at least d-2, and so on up to antipodal vertices, whose colors are permitted to be the same. For this reason, (d-1) radio colorings are also referred to as antipodal colorings.

In the case of an antipodal coloring of the path P_n of order $n \geq 2$, only the two end-vertices are permitted to be colored the same. If u and v are distinct vertices of P_n and d (u, v) = i, then $|c(u)-c(v)| \geq n-1-i$. Since P_n is a tree, not only is i the length of a shortest u – u path in P_n, it is the length of the only u – v path in P_n. In particular, is the length of a longest u – v path?

The detour distance D (u, v) between two vertices u and v in a connected graph G is defined as the length of a longest u – v path in G. Hence the length of a longest u – v path in P_n is D (u, v) = d (u, v). Therefore, in the case of the path P_n, an antipodal coloring of P_n can also be defined as a vertex coloring c that satisfies.

$D(u,v) + |c(v)| \geq n - 1$, for every two distinct vertices u and v of P_n.

§1.1 Definition: Vertex coloring c that satisfy were extended from paths of order n

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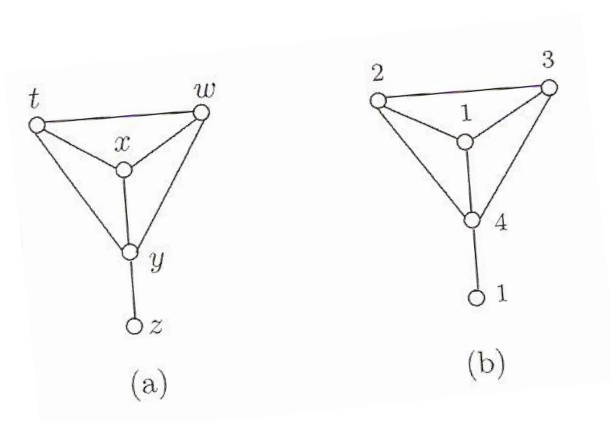
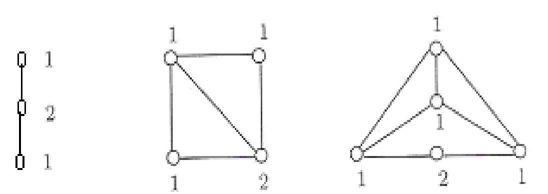
to arbitrary connected graphs of order n by Gary Chartrand, Ladislav Nebesky, and Ping Zhang .A **Hamiltonian coloring** of a connected graph G of order n is a vertex coloring c such that, **D (u,v) + |c(v)| $\geq$ n – 1,** for every tow distinct vertices u and v of G. the largest color assigned to a vertex of G by c is called the **value** of c and is denoted by hc(c). The **Hamiltonian chromatic number** hc (G) is the smallest value among all Hamiltonian colorings of G.

EX: Figure.1 (a) shows a graph H of order 5. A vertex coloring c of H is shown in Figure.1 (b). Since D (u, v) + |c (u) - c (v)| $\geq$ 4 for every two distinct vertices u and u of H, it follows that c is a Hamiltonian coloring and so hc(c) =4. Hence hc (H) $\geq$ 4. Because no two of the vertices t, w, x, and y are connected by a Hamiltonian path, these must be assigned distinct colors and so hc (H) $\geq$ 4. Thus hc (H) = 4.



1. A graph with Hamiltonian chromatic number 4

If a connected graph G of order n has Hamiltonian chromatic number 1, then D(u,v) = n – 1 for every two distinct vertices u and v of G and consequently G is Hamiltonian-connected, that is, every two vertices of G are connected by a Hamiltonian pat. Indeed, hc (G) = 1 if and only if G is Hamiltonian – connected. Therefore, the Hamiltonian Chromatic Number of a connected graph G can be considered as a measure of how close G is to being Hamiltonian connected. That is the closer hc (G) is to 1, the closer G is to being Hamiltonian connected. The three graphs H_1, H2 and H_3 shown in below figure 2 are all close (in this sense) to being Hamiltonian-connected since hc (H_i) = 2 for i = 1, 2, 3.



H_1 H_2

H_3

2. Three graphs with Hamiltonian chromatic number 2

§2 Theorem: For every integer $n \geq 3$, hc $(K_{1,n-1}) = (n-2)^2 + 1$

Proof: Since hc $(K_{1,2}) = 2$ (See H_1 in Figure 2) we may assume that $n \geq 4$.

Let $G = K_{1,n-1}$ where $V(G) = \{v_1, v_2 \ldots.. v_n\}$ and v_n is the central vertex. Define the coloring c of G by $c(v_n) = 1$ and $C(v_i) = (n-1) + (i-1)(n-3)$ for $1 \leq i \leq n-1$.Then c is a Hamiltonian coloring of G and

$hc(G) \leq hc(c) = c(v_{n-1}) = (n-1) + (n-2)(n-3) = (n-2)^2 + 1$.It remains to show that hc (G) $\geq (n-2)^2 + 1$.

Let c be a Hamiltonian coloring of G such that hc(c) = hc (G). Because G contains no Hamiltonian path, c assigns distinct colors to the vertices of G. We may assume that $C(v_1) < c(v_2) < \ldots < c(v_{n-1})$. We now consider three cases, depending on the color assigned to the central vertex v_n.

Case 1.

$$c(v_n) = 1.$$

Since

$$D(v_1, v_n) = 1 \text{ and } D(v_i, v_{i+1}) = 2 \text{ for } 1 \leq i \leq n-2.$$

It follows that

$$C(v_{n-1}) \geq 1 + (n-2) + (n-2)(n-3) = (n-2)^2 + 1$$

And so

$$\underline{hc}(G) = hc(c) = c(v_{n-1}) \geq (n-2)^2 + 1.$$

Case 2.

$$C(v_n) = hc(c)$$

Thus, in this case,

$$1 = c(v_1) < c(v_2) < \ldots < c(v_{n-1}) < c(v_n)$$

Hence

$$C(v_n) \geq 1 + (n-2)(n-3) + (n-2) = (n-2)^2 + 1$$

And so

$$hc(G) = hc(v) = c(v_n) \geq (n-2)^2 + 1.$$

Case 3.

$$C(v_j) < c(v_n) < c(v_{j+1}) \text{ for some integer } j \text{ with } 1 \leq j \leq n-2.$$

Thus $c(v_1) = 1$ and $c(v_{n-1}) = hc(c)$.

in this case

$$C(V_j) \geq 1 + (j-1)(n-3),$$
$$C(vn) \geq c(v_j) + (n-2)$$
$$C(v_{j+1}) \geq c(v_n) + (n-2) \text{ and}$$
$$C(v_{n-1}) \geq c(v_{j+1}) + [(n-1) - (j+1)](n-3).$$

Therefore,

$$C(v_{n-1}) \geq 1 + (j-1)(n-3) + 2(n-2) + (n-j-2)(n-3)$$
$$= (2n-3) + (n-3)^2 = (n-2)^2 + 2 > (n-2)^2 + 1.$$

And so $hc(G) = hc(c) = c(v_{n-1}) > (n-2)^{2+}1$.

Hence in any case,

$$hc(G) \geq (n-2)^2 + 1 \text{ and so } hc(G) = (n-2)^2 + 1.$$

§ 3 Theorem: For every integer $n \geq 3$, hc (Cn) = n-2.

Proof.

Since we noted that hc (C_n) = n-2 for n = 3, 4, 5. We may assume that $n \geq 6$. Let $C_n = (v_1, v_2 \ldots v_n , v_1)$. Because the vertex coloring c of C_n defined by $c(v_1) = c(v_2) = 1$, $c(v_{n-1}) = c(v_n) = n-2$ and $c(v_i) = i-1$ for $3 \leq i \leq n-2$ is a Hamiltonian coloring, it follows that $hc(C_n) \leq n-2$.Assume, to the contrary , that hc $(C_n) < n-2$ for some integer $n \geq 6$. Then there exists a Hamiltonian (n-3) coloring c of C_n. We consider two cases, according to whether n is odd or n is even.

Case 1.

n is odd: Then n = 2k + 1 for some integer $k \geq 3$. Hence there exists a Hamiltonian (2k-2) coloring c of C_n. Let,

A = {1, 2… k-1} and B = {k, k+1…2k-2}

For every vertex u of C_n, there are two vertices v of C_n such that D (u,v) is minimum (and d(u,v) is maximum), namely D(u,v) = d(u,v) + 1 = k+1. For u = vi, these two vertices v are v_{i+k} and v_{i+k+1} (where the addition in i + k and i + k + 1 is performed modulo n).Since c is a Hamiltonian coloring. $D (u, v) + |c (u) – c (v)| \geq n = l = 2k$.BecauseD (u, v) = k + 1, it follows that

$|c(u) – c(v)| \geq k – 1$.

Therefore, if $c (u) \in A$, then the colors of these two vertices v with this property must belong to B. In particular, if $c (v_i) \in A$, then $(v_{i+k}) \in B$. Suppose that there are a vertices of C_n whose colors belong to A and b vertices of C_n whose colors belong to B. Then $b \geq a$ However, if $c (v_i) \in B$, then c (v_{i+k}) belongs to a implying that $a \geq b$ and so. a=b. since a + b = n and n is odd, this is impossible.

Case 2.

n is even: Then n = 2k for some integer $k \geq 3$. Hence there exists a Hamiltonian (2k-3) - coloring c of C_n. For every vertex u of C_n, there is a unique vertex v of C_n for which D (u, v) is minimum (and d (u, v) is maximum), namely, d (u, v) = k. For $u = v_i$, this vertex v is v_{i+k} (where the addition in i + k is performed modulo n).

Since c is a Hamiltonian coloring, $D (u,v) + |c(u) – c(v)| \geq n – 1 = 2k – 1$. Because D (u, v) = k, it follows that $|c (u) – c (v)| \geq k – 1$. This implies, however, that if

c (u) = k-1, then there is no color that can be assigned to u to satisfy this requirement. Hence no vertex of C_n can be assigned the color k- 1 by c.

Let, A = {1, 2… k-2} and B = {k, k+1…2k-3}.

Thus |A| = |B| = k – 2. If c (v_i) ∈ A, then c (v_{i+k}) ∈ B. Also, if c (v_i) ∈ B, then c (v_{i+k}) ∈ A. Hence there are k vertices of C_n assigned colors from B.Consider two adjacent vertices of C_n, one of which is assigned a color from A and the other is assigned a color from B. We may assume that c (v_1) ∈ A and c (v_2) ∈ B. Then c (v_{k+1}) ∈ B. Since D (v_2,v_{k+1}) = k+1, it follows that |c (v_2) – c(v_{k+1}) ≥ k – 2. Because c (v_2), c (v_{k+1}) ∈ B, this implies that one of c (v_2) and c (v_{k+1}) is at least 2k-2. This is a contradiction.

§ 3.1Proposition: If H is a spanning connected sub graph of a graph G, then hc (G) ≤ hc (H)

Proof.

Suppose that the order of H is n. Let c be a Hamiltonian coloring of H such that hc(c) hc (H). Then D_H (u,v) + |c(u) – c(v)| ≥ n -1 for every two distinct vertices u and v of H. since D_G (u,v) ≥ D_H (u,v) for every two distinct vertices u and v of H, it follows that D_G (u,v) + |c(u) – c(v)| ≥ n – 1 and so c is a Hamiltonian coloring of G as well. Hence hc (G) ≤ hc (c) = hc (H).

§ 3.2 Proposition: Let H be a Hamiltonian graph of order n – 1 ≥ 3. If G is a graph obtained by adding a pendant edge to H, then he (G) = n – 1.

Proof. Suppose that C = (v_1,v_2…v_{n-1},v_1) is a Hamiltonian cycle of H and v_1v_n is the pendant edge of G. Let c be a Hamiltonian coloring of G. Since D_G (u, v) ≤ n-2 for every two distinct vertices u and v of C, no two vertices of C can be assigned the same color by c. Consequently, hc (c) > n – 1 and so hc (G) ≥ n – 1.

Now define a coloring c` of G by

$$C^1(v_i) = \begin{cases} i & \text{if } 1 < i < n-1 \\ n-1 & \text{if } i = n. \end{cases}$$

We claim that c` is a Hamiltonian coloring of G. First let v_j and v_k be two vertices of C where 1 ≤ j < k ≤ n – 1. The $|c^1 + (v_j) - c^1(v_k)| = k - j$ and

D (v_j,v_k) = max {k-j, (n-1) – (k-j)}.

In either case, D (v_j,v_k) ≥ n-1 + j – k and so

$$D(v_j,v_k) + |c^1(v_j) - c^1(v_k)| \geq n-1.$$

For 1≤ j ≤n-1, $|c^1(v_j) - c^1)v_n)| = n-1-j$, while

D (v_j,v_n) ≥ max {j, n-j+1}

And so, D (v_j, v_n) ≥ j.

Therefore,

$$D(v_j, v_n) + |c^1(v_j) - c^1(v_n)| \geq n-1.$$

Hence, as claimed, c’ is a Hamiltonian coloring of G and so

$$hc(G) \leq hc(c') = c^1(v_n) = n-1.$$

§4 Theorem: for every connected graph G of order n ≥ 2,

hc (G) ≤ $(n-2)^2 + 1$.

Proof. First, if G contains a vertex of degree n-1, then G contains the star $K_{1,n-1}$ as a spanning sub graph. Since hc $(K_{1,n-1}) = (n-2)^2 + 1$ it follows by proposition 1 that hc(G) ≤ $(n-2)^2 + 1$. Hence we may assume that G contains a spanning tree T that is not a star and so its complement T contains a Hamiltonian path P = $(v_1,v_2....v_n)$. Thus $v_i v_{i+1} \notin E$ (T) for $1 \leq i \leq n-1$ and so $D_T (v_i,v_{i+I}) \geq 2$. Define a vertex coloring c of T by

$$C(v_i) = (n-2) + (i-2)(n-3) \text{ for } 1 \leq i \leq n.$$

Hence

$$hc(c) = c(v_n) = (n-2) + (n-2)(n-3) = (n-2)^2$$

Therefore, for integers i and j with $1 \leq i < j \leq n$,

$$|c(v_i) - c(v_j)| = (j-i)(n-3).$$

If $j = i+1$, then

$$D(v_i,v_j) + (c(v_i) - c(v_j)| \geq 1 + 2(n-3) = 2n-5 \geq n-1.$$

Thus c is a Hamiltonian coloring of T. therefore,

$$hc(G) \leq hc(T) \leq hc(c) = c(v_n) = (n-2)^2 < (n-2)^2 + 1,$$

Which completes the proof

REFERENCES;

1. G.Chartrand, L.Nebesky, and P.Zhang, Hamiltonian graphs. Discrete Appl.Math. 146(2005) 257-272.
2. O.Ore Note on Hamalton circuits.Amer.Math.Monthly 67(1960)
3. P.G Tait, Remarks on the colorings of maps proc.Royal soc.Edinburgh 10 (1880)
4. H.Whitney, the Colorings of Graphs.Ann.Math.33 (1932) 688-718.
5. D.R.Woodall, List colourings of graphs. Survey in combinatories 2001 Cambridge univ. press, Cambridge,(2001) 269-301.