Equilibrium of Stars within the Framework of Generalized Special Relativity Theory

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Abstract:
Generalized Special Relativistic energy expression, beside Fermi momentum and ordinary Newtonian gravity potential are used for stars equilibrium conditions. The radius which makes the energy minimum shows that stability requires the mass to be less than certain critical mass which reflects quantum gravity behavior. This condition is similar to that to General Relativity. The radius should be greater than certain critical value. This critical value is typical to that of general relativity for black hole. The equilibrium conditions show that pressure and centrifugal force should counter balance attractive gravity force. It also shows that kinetic energy balancer potential energy at equilibrium. The mathematical model is simple compared to general relativity model.

Keywords: equilibrium, stars, generalized special relativity, gravity potential, centrifugal force, pressure, critical mass, critical radius.

1. Introduction:
A main-sequence hydrogen-burning star, such as the Sun, is maintained in equilibrium via the balance of the gravitational attraction tending to make it collapse, and the thermal pressure tending to make it expand of course, the thermal energy of the star is generated by nuclear reactions occurring deep inside its core. Eventually, however, the star will run out of burnable fuel, and, therefore, start to collapse, as it radiates away its remaining thermal energy [1]. What is the ultimate fate of such a star? A burnt-out star is basically a gas of electrons and ions. As the star collapses, its density increases, so the mean separation between its constituent particles decreases. Eventually, the mean separation becomes of order wavelength of the electrons, and the electron gas becomes degenerate. Note, that the wavelength of the ions is much smaller than that of the electrons, so the ion gas remains non-degenerate. Now, even at zero temperature, a degenerate electron gas exerts a substantial pressure, because the Pauli exclusion principle prevents the mean electron separation from becoming significantly smaller than the typical wavelength of the electrons.Thus, it is

possible for a burnt-out star to maintain itself against complete collapse under gravity via the degeneracy pressure of its constituent electrons. Such stars are termed white-dwarfs. At stellar densities which greatly exceed white-dwarf densities, the extreme pressures cause electrons to combine with protons to form neutrons [2]. Thus, any star which collapses to such an extent that its radius becomes significantly less than that characteristic of a white-dwarf is effectively transformed into a gas of neutrons. Eventually, the mean separation between the neutrons becomes comparable with their wavelength. At this point, it is possible for the degeneracy pressure of the neutrons to halt the collapse of the star [3]. A star which is maintained against gravity in this manner is called a neutron star.it is found that there is a critical mass and critical radius above which a neutron star cannot be maintained against gravity. this critical radius, which is known as the radius of Schwarzschild. A star whose radius exceeds the radius of Schwarzschild [4,5] cannot be maintained against gravity by degeneracy pressure, and must ultimately collapse to form a black hole.One will discuss in this research the equilibrium of stars by pressure and gravity forces within the Framework of generalized special relativity, in section 2. Sections 3 and 4 are concerned with discussion and conclusion respectively.

2. Equilibrium Conditions:

Consider first the Generalized Special Relativity GSR energy E equilibrium condition by minimizing E w.r.t. r

$$E = m_0c^2\left(1+\frac{2\varphi}{c^2}\right)\left(1+\frac{2\varphi}{c^2}-\frac{v^2}{c^2}\right)^{-1/2} \quad (1)$$

$$\varphi = -\frac{GM}{r} \quad , \quad m_0 = M \quad (2)$$

$$\frac{v^2}{c^2} = \frac{m^2v^2}{m^2c^2} = \frac{p^2}{m^2c^2} = \frac{p^2}{M^2c^2} \quad (3)$$

For simplicity consider the average momentum p is equal to the maximum momentump_F, by ignoring $\sqrt{2}$, where

$$p = \frac{p_F}{\sqrt{2}}$$

Thus

$$p = p_F = \Lambda\left(\frac{N}{V}\right)^{\frac{1}{3}} = \Lambda n_0$$

Where

$$\Lambda = (3\pi^2)^{\frac{1}{3}}\hbar \quad (4)$$

Therefore, with the aid of equations (2) – (4), equation (1) reads

$$E = E_F = Mc^2\left(1-\frac{2MG}{r}\right)\left(1-\frac{2MG}{r}-\frac{{p_F}^2}{M^2c^2}\right)^{-\frac{1}{2}} \quad (5)$$

The radius r which makes the energy E minimum is given when

$$\frac{dE_r}{dr}=\frac{Mc^2\left(\frac{2MG}{r^2}\right)}{\left(1-\frac{2MG}{r}-\frac{p_F{}^2}{M^2c^2}\right)^{1/2}}+\frac{Mc^2\left(1-\frac{2MG}{r}\right)\left(-\frac{1}{2}\right)\left(\frac{2MG}{r^2}\right)}{\left(1-\frac{2MG}{r}-\frac{p_F{}^2}{M^2c^2}\right)^{3/2}}=0$$

$$\frac{Mc^2\left[\left(\frac{2MG}{r^2}\right)\left(1-\frac{2MG}{r}-\frac{p_F{}^2}{M^2c^2}\right)-\left(\frac{MG}{r^2}\right)\left(1-\frac{2MG}{r}\right)\right]}{\left(1-\frac{2MG}{r}-\frac{p_F{}^2}{M^2c^2}\right)^{3/2}}=0$$

$$\frac{M^2c^2G}{r^2}\left(-1+2+\frac{4MG}{r}-\frac{p_F{}^2}{M^2c^2}\right)=0 \quad (6)$$

This is satisfied when

$$\frac{4MG}{r}=\frac{p_F{}^2}{M^2c^2}-1 \quad (7)$$

Thus the minimum radius is given by

$$r=\frac{4M^3c^2G}{p_F{}^2-M^2c^2} \quad (8)$$

Where

$$p_F=(3\pi^2)^{1/3}\hbar n^{1/3}=\Lambda\left(\frac{N}{V}\right)^{1/3}=\left(\frac{9\pi}{4}\right)^{1/3}\frac{N^{1/3}}{r_F}\hbar \quad (9)$$

The equilibrium takes place when r is non negative, i.e when

$$p_F{}^2>M^2c^2$$

$$p_F>\ Mc \quad (10)$$

The critical mass is given by

$$M_c=\frac{p_F}{c} \quad (11)$$

Thus for star to be at equilibrium one requires

$$\frac{p_F}{c}>M$$

$$M_C>M \quad (12)$$

$$M<M_C$$

Thus the maximum mass for stable star is

$$M_c=\frac{p_F}{c}=\frac{(3\pi^2)^{1/3}\hbar}{c}\left(\frac{N}{V}\right)^{1/3} \quad (13)$$

This condition resembles Chandrasekhar limit for stable white dwarf.
i.e the star mass need to be less than the critical value in equation (11).
The equilibrium condition can also be found by using generalized special relativity energy momentum relation

$$g_{00}\,E^2-p^2c^2=m_0^2c^4(g_{00})^2$$

$$E^2=(g_{00})^{-1}p^2c^2+g_{00}\,m_0^2c^4 \quad (14)$$

One can rewrite equation (14) to be

$$E = (a_1 - a_2 p^2)^{1/2} \qquad (15)$$

Where

$$a_1 = g_{00} m_0^2 c^4 = \left(1 - \frac{2MG}{rc^2}\right) m_0^2 c^4 \quad , \quad a_2 = (g_{00})^{-1} c^2$$

$$a_2 p^2 = a_1 cos^2\theta \qquad (16)$$

$$E = \int_0^{p_F} (a_1 - a_1 cos^2\theta)^{1/2} dp \; (17)$$

Where

$$-dp = \sqrt{\frac{a_1}{a_2}}\, sin\theta d\theta \qquad (18)$$

$$E = \sqrt{a_1} \int (1 - cos^2\theta)^{1/2} \left(-\sqrt{\frac{a_1}{a_2}}\right) sin\,\theta\, d\theta \qquad (19)$$

$$= \sqrt{a_1}\left(-\sqrt{\frac{a_1}{a_2}}\right) \int \sin^2\theta\, d\theta \qquad (20)$$

$$cos2\theta = cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$E = \frac{\sqrt{a_1}}{2}\left(-\sqrt{\frac{a_1}{a_2}}\right)\left(\theta - \frac{sin2\theta}{2}\right) \qquad (21)$$

$$sin2\theta = 2sin\theta cos\theta \quad , \quad cos\theta = \sqrt{\frac{a_2}{a_1}}\, p$$

$$sin\theta = (1 - \cos^2\theta)^{1/2} = \left(1 - \frac{a_2}{a_1} p^2\right)^{1/2}$$

$$E = \sqrt{a_1} \sqrt{\frac{a_1}{a_2}} \sqrt{\frac{a_2}{a_1}} p_F (1 - a_3 p_F^2)^{1/2} + cos^{-1} \sqrt{\frac{a_2}{a_1}} p_F - \frac{\pi}{2}$$

$$= \sqrt{a_1} p_F (1 - a_3 p_F^2)^{1/2} + cos^{-1} \sqrt{\frac{a_2}{a_1}} p_F - \frac{\pi}{2} \qquad (22)$$

Where

$$a_3 = \frac{a_2}{a_1} = \frac{g_{00}^{-1} c^2}{g_{00} m_0^2 c^4} = \frac{g_{00}^{-2}}{m_0^2 c^2}$$

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$$E = \left(1 - \frac{2MG}{rc^2}\right)^{\frac{1}{2}} p_F \left[1 - \frac{p_F^2\, c^2}{m_0^2 c^4 \left(1 - \frac{2MG}{rc^2}\right)^2}\right]^{1/2} + cos^{-1}\left(\frac{p_F c}{m_0 c^2 \left(1 - \frac{2MG}{rc^2}\right)}\right) - \frac{\pi}{2} \quad (23)$$

It is clear from equation (23) that stability requires E to be real.This can be satisfied when

$$1 - \frac{2MG}{rc^2} > \quad 0$$

$$rc^2 > \quad 2MG$$

$$r > \frac{2MG}{c^2}$$

The critical radius is given by

$$r_c \; = \; \frac{2MG}{c^2} \qquad (24)$$

Thus the radius should be greater than the black hole radius. Also

$$1 - \frac{p_F^2\, c^2}{m_0^2 c^4 \left(1 - \frac{2MG}{rc^2}\right)^2} > \; 0$$

$$m_0^2 c^4 \left(1 - \frac{2MG}{rc^2}\right)^2 > p_F^2\, c^2$$

Thus

$$m_0 c^2 \left(1 - \frac{2MG}{rc^2}\right) > \; \pm p_F c$$

$$\left(1 - \frac{2MG}{rc^2}\right) > \quad \pm \frac{p_F c}{m_0 c^2}$$

$$rc^2 - 2MG \quad > \; \pm \left(\frac{p_F c}{m_0 c^2}\right) rc^2 \qquad (25)$$

$$\left(1 \pm \frac{p_F c}{m_0 c^2}\right) rc^2 > \quad 2MG$$

$$r \quad > \frac{2MGm_0}{(m_0 c^2 \pm p_F c)} \qquad (26)$$

Thus the critical radius is given by

$$r_c = \frac{2Mm_0 G}{(m_0 c^2 \pm p_F c)} \qquad (27)$$

The equilibrium mass also satisfies

$$2MG \quad > \; -rc^2 \pm \left(\frac{p_F c}{m_0 c^2}\right) rc^2$$

$$M \quad < \frac{rc^2}{2G} \pm \frac{p_F rc}{m_0} \qquad (28)$$

Hence the critical maximum mass is given by

$$M_c = \frac{rc^2}{2G} \pm \frac{p_F rc}{m_0} \tag{29}$$

The equilibrium condition can also be found by minimizing E, where

$$E = mc^2 = m_0 c^2 \left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{30}$$

Assuming the mass to be equal to the rest mass, and the potential to be the Newtonian, one gets

$$m_0 = M \qquad , \qquad \varphi = -\frac{GM}{R} \tag{31}$$

Therefore

$$E = Mc^2 \left(1 - \frac{2GM}{Rc^2}\right)\left(1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{32}$$

For small φ and velocity v compared to speed of light c, i.e

$$\frac{GM}{R} < 1 \quad , \quad \frac{v^2}{c^2} < 1$$

One gets

$$E = Mc^2 \left(1 - \frac{2GM}{Rc^2}\right)\left(1 + \frac{GM}{Rc^2} + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$E = \left(Mc^2 - \frac{2GM^2}{R}\right)\left(1 + \frac{GM}{Rc^2} + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$E = Mc^2 + \frac{GM^2}{R} + \frac{1}{2}Mv^2 - \frac{2GM^2}{R} - \frac{2G^2M^3}{R^2c^2} - \frac{GM^2v^2}{Rc^2} \tag{33}$$

The mass which make the energy minimum for constant radius is given by

$$\frac{dE}{dM} = \frac{c^2\left(1 - \frac{2GM}{Rc^2}\right)}{\sqrt{1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}}} + \frac{Mc^2\left(\frac{-2G}{Rc^2}\right)}{\sqrt{1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}}} + \frac{\frac{1}{2}Mc^2\left(1 - \frac{2GM}{Rc^2}\right)\left(\frac{2G}{Rc^2}\right)}{\left(1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}\right)^{3/2}} \tag{34}$$

Neglecting the kinetic term yields

$$\frac{dE}{dM} = \frac{\left(c^2 - \frac{4MG}{R}\right)\left(1 - \frac{2MG}{Rc^2}\right) + \frac{MG}{R} - \frac{2M^2G^2}{R^2c^2}}{\left(1 - \frac{GM}{Rc^2} + \frac{1}{2}\frac{v^2}{c^2}\right)^{3/2}} = 0 \tag{35}$$

This requires

$$c^2 - \frac{2MG}{R} - \frac{4MG}{R} + \frac{8M^2G^2}{R^2c^2} + \frac{MG}{R} - \frac{2M^2G^2}{R^2c^2} = 0$$

$$c^2 - \frac{5MG}{R} + \frac{6M^2G^2}{R^2c^2} = 0$$

$$\frac{6G^2}{R^2c^2}M^2 - \frac{5G}{R}M + c^2 = 0$$

$$ax^2 + bx + c = 0 \ , \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$M = \frac{\frac{5G}{R} \pm \sqrt{\left(\frac{5G}{R}\right)^2 - \frac{24G^2c^2}{R^2c^2}}}{\frac{12G^2}{R^2c^2}} = \frac{R^2c^2}{12G^2}\left(\frac{5G}{R} \pm \sqrt{\frac{G^2}{R^2}}\right)$$

$$M = \frac{R^2c^2}{12G^2}\left(\frac{G}{R}\right)(5 \pm 1) = \frac{Rc^2}{12G}(5 \pm 1)$$

$$M = \frac{1}{2}\frac{Rc^2}{G} \quad , \qquad \frac{1}{3}\frac{Rc^2}{G} \qquad (36)$$

For stars one have two forces, pressure force which counter balance the gravity force, thus

$$P = \frac{NKT}{V} = \frac{1}{3}\frac{mv^2}{V} \qquad (37)$$

Thus the pressure force is given by

$$F_P = PA = \frac{\frac{1}{3}mv^2(4\pi r^2)}{\frac{4\pi}{3}r^3} = \frac{mv^2}{r} \qquad (38)$$

The gravity force is given by

$$F_g = \frac{GmM}{r^2} \qquad (39)$$

At equilibrium the two forces counter balances them selves thus

$$F_P = F_g$$

$$\frac{mv^2}{r} = \frac{GmM}{r^2} \quad , \qquad mv^2 = \frac{GmM}{r} \qquad (40)$$

If particles are considered as strings with v representing max speed. thus the average value is given by

$$v_a = \frac{v_m}{\sqrt{2}} \quad , \qquad mv_a^2 = \frac{mv_m^2}{2} \qquad (41)$$

Thus

$$mv_a^2 = \frac{mv_m^2}{2} = \frac{1}{2}mv^2 \qquad (42)$$

One thus gets

$$\frac{1}{2}mv^2 = \frac{GmM}{r} = m\varphi \qquad (43)$$

Hence

$$v^2 = 2\varphi \qquad (44)$$

Hence

$$E = \frac{m_0c^2\left(1 + \frac{2\varphi}{c^2}\right)}{\left(1 + \frac{2\varphi - v^2}{c^2}\right)^{1/2}} = m_0c^2\left(1 + \frac{2\varphi}{c^2}\right) \qquad (45)$$

But

$$m_0 = M \quad , \quad \varphi = -\frac{GM}{R} \tag{46}$$

For attractive force

$$E = M\left(c^2 - \frac{2GM}{R}\right) \tag{47}$$

$$\frac{dE}{dM} = \left(c^2 - \frac{2GM}{R}\right) + M\left(\frac{-2G}{R}\right) = 0$$

$$-\frac{4GM}{R} + c^2 = 0$$

$$\frac{4GM}{R} = c^2 \tag{48}$$

$$M = \frac{Rc^2}{4G} \tag{49}$$

$$M = \frac{R}{2G}c_a^2 = \frac{R}{2G}\left(\frac{c_m}{\sqrt{2}}\right)^2 = \frac{R}{2G}c^2 \tag{50}$$

For

$$c \;\rightarrow\; c_a = \frac{c_m}{\sqrt{2}} = \frac{c}{\sqrt{2}}$$

3. Discussion:

In this work Generalized Special Relativity energy relation (1) is used by assuming the average momentum p to be related to the maximum momentum p_F, beside the ordinary expression for Newton gravity potential (see (2), (3) and (4)) to get the expression for E. The radius which make E minimum in (8) requires maximum mass given by (13). The condition for maximum mass resembles Chandrasekhar Limit. However the expression does not depend on the gravitational constant G.Using the same steps used in the conventional General Relativity theory, E is integrated over the momentum p. A useful expression for E was found in (23). The equilibrium condition requires, here, E to be real. This makes the critical radius to be dependent on G and h as shown by equations (27) and (9). Equation (26) shows that this is the minimum radius which secures equilibrium.But according to equations (28) and (29) the maximum critical mass depends two on G and h. the pence of these tow parameters reflects the quantum gravitational nature of the steller mass.The equilibrium condition is also studied by considering the effect of pressure force in relation to centrifugal force. Surprisingly equations (38) shows that the pressure force act as a centrifugal force which counter balance the gravity force. By treating particles as strings it was shown by equation (37) that equilibrium takes place when kinetic and potential energy balances each other. The mass which makes E minimum also tackled in equations (34, 35 and 36). The mass at which E is minimum is given by equation conforms with that of black hole (see equation (50))

4. Conclusion:

The GeneralizedSpecial Relativity can successfully describe stars equilibrium condition. It shows that the equilibrium conditions are related to certain critical mass and radius values, beside effects of pressure, centrifugal force and attractive gravity force, similar to that obtained by General Relativity.

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