IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 4 Issue 2, February 2017
ISSN (Online) 2348 – 7968 | Impact Factor (2016) – 5.264
www.ijiset.com

Optimal Control and Problem with Linear Quadratic Cost Function

Habib Molaei
Department of Mathematics Technical and Vocational, University Gazi-Tabatabaei Urmia, Iran, P.O. Box 57169-33950

Abstract
In this paper, we explore exactly this approach for the linear quadratic optimal control problem. We will consider how to stabilize linear system by using feedback control. This topic is very useful in physics, Biology and Engineering. We consider the linear Quadratic optimal control and introduce the riccati differential equation and riccati algebraic equation. We look at the optimum of a function and functional and some variational problems. Various steps used in finding the optimal solution to these variational problems discussed. We also consider the exterma of functions and functionals with conditions. Also, Examples on finding the exterma were given with their solutions.

Mathematics subject classification: 49J15, 49K21, 34B15

Keywords: optimal control, Boundary conditions, Lagrangian function, Linear quadratic, Cost function

1. Introduction

Consider minimization of a general performance index for an arbitrary initial point (x,t)

$$\mathrm{J}(\mathrm{u}) = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} \mathrm{V}(x(\mathrm{t}), \mathrm{u}(\mathrm{t}), \mathrm{t})\mathrm{dt} \tag{1}$$

Subject to the following system with final time constraints:

$$\dot{x}(t) = \mathrm{F}(x(\mathrm{t}), \mathrm{u}(\mathrm{t}), \mathrm{t}) \quad , \quad \mathrm{t} \epsilon [t_0, t_f] \tag{2}$$

$$x(\mathrm{t}_0) = x_0 \tag{3}$$

$$x(\mathrm{t}_0), t_f \text{ is free} \tag{4}$$

Here, $x \epsilon \mathrm{R}^n$, $\mathrm{t}\epsilon\mathrm{R}$, $\mathrm{u}\epsilon\mathrm{R}^m$ and φ is functional of x(t),V is a functional of x(t),u(t)

We assume that there exists no constraints on the state and control trajectiories. Our objective is to evaluate the optimal trajectory statisfying the final time constrains and to find the optimal feedback control for an arbitrary initial point $(x,\mathrm{t})\epsilon\mathrm{R}^{n+1}$.

We consider two representative problem formulations, which are characterized by the types of terminal boundary conditions:

i. Hard constraint problem (HCP): $\varphi(x(t_f), t_f) = 0$, $\varphi(x(t_f), t_f) = x(t_f) - x_f$.
ii. Soft constraint problem (SCP): $\varphi(x(t_f), t_f)$ dose not exit and $\varphi(x(t_f), t_f)\epsilon\mathrm{R}$.

2. Variational approach to optimal control system

Equivalent to the a love equation is as follows:
Find the extrema of

$$\mathrm{J}(\mathrm{u}) = \int_{\mathrm{t}_0}^{t_f} \left[\mathrm{V}(x(\mathrm{t}), \mathrm{u}(\mathrm{t}), \mathrm{t}) + \frac{\mathrm{d}\varphi}{\mathrm{dt}}\right] \mathrm{dt} \tag{5}$$

Subject to

$$\dot{x}(t) = f(x(t), u(t), t) \tag{6}$$

$$x(\mathrm{t}_0) = x_0 \tag{7}$$

$$x(t_f), t_f \text{ is free} \tag{8}$$

Solution

Step1: Introduce the Lagrangian function L i.e

$$\mathrm{L}(x, \dot{x}, u, \lambda, t) = \left[\mathrm{V}(x(\mathrm{t}), \mathrm{u}(\mathrm{t}), \mathrm{t}) + \frac{\mathrm{d}\varphi}{\mathrm{dt}}\right] = \lambda(\mathrm{t})(\mathrm{f}(x(\mathrm{t}), \mathrm{u}(\mathrm{t}), \mathrm{t}) - \dot{x}(t)) \tag{9}$$

Also introduce

$$J_a = J_a(x,\dot{x},u,\lambda,t_f) = \int_{t_0}^{t_f} L(x,\dot{x},u,\lambda,t)dt \tag{10}$$

Step2: Find the first variation given increments $\Delta x, \Delta\dot{x}, \Delta u, \Delta t_f$

$$\frac{\partial J_a}{\partial x}=0,\ \frac{\partial J_a}{\partial \dot{x}}=0,\ \frac{\partial J_a}{\partial u}=0,\ \frac{\partial J_a}{\partial \lambda}=0 \tag{11}$$

$$J_a(x+\Delta x, \dot{x}+\Delta\dot{x}, u+\Delta u, t_f+\lambda t_f) - J_a(x,\dot{x},u,t_f) =$$
$$\int_{t_0}^{t_f+\Delta t_f} L(x+\Delta x, \dot{x}+\Delta\dot{x}, u+\Delta u, \lambda)dt - \int_{t_0}^{t_f} L(x,\dot{x},u,\lambda,t)dt =$$
$$\int_{t_0}^{t_f}[L(x+\Delta x,\dot{x}+\Delta\dot{x},u+\Delta u,\lambda) - L(x,\dot{x},u,\lambda)]dt + \int_{t_f}^{t_f+\Delta t_f} L(x+\Delta x,\dot{x}+\Delta\dot{x},u+\Delta u,\lambda)dt \tag{12}$$

By Taylor series expansion

$$= \int_{t_0}^{t_f}\left(\frac{\partial l}{\partial x}\Delta x + \frac{\partial l}{\partial \dot{x}}\Delta\dot{x} + \frac{\partial l}{\partial u}\Delta u + oh^{(p)}\right)dt + L(x,\dot{x},u,\lambda)\Big|_{t=t_f}^{\Delta t_f} + oh^{(p)} \tag{13}$$

Using integration by parts

$$\int_{t_0}^{t_f}\frac{\partial l}{\partial \dot{x}}\Delta\dot{x}\,dt = \int_{t_0}^{t_f}\frac{\partial l}{\partial \dot{x}}d(\Delta x) = \frac{\partial l}{\partial \dot{x}}\Delta x\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f}(\Delta x)\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right)dt \tag{14}$$

Equation (13) becomes

$$= \int_{t_0}^{t_f}\left(\frac{\partial l}{\partial x} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right)\right)\Delta x dt + \int_{t_0}^{t_f}\frac{\partial l}{\partial u}\Delta u\,dt + L(x,\dot{x},u,\lambda)\Big|_{t=t_f}^{\Delta t_f} + \frac{\partial l}{\partial \dot{x}}\Delta x\Big|_{t_0}^{t_f} + oh^{(p)} \tag{15}$$

The necessary condition for the extrema point are

$$\frac{\partial l}{\partial x} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right) = 0 \tag{16}$$
$$\frac{\partial l}{\partial u} = 0$$
$$L(x,\dot{x},u,\lambda)\Big|_{t=t_f}^{\Delta t_f} + \left(\frac{\partial l}{\partial \dot{x}}\Delta x\right)\Big|_{t_0}^{t_f} = 0$$
$$\frac{\partial l}{\partial \lambda} = 0$$

Step3: Introduce Hamiltonian function $H(x,u,\lambda,t)$

$$H(x,u,\lambda,t) = V(x,u,t) + \lambda(t)f(x,u,t) \tag{17}$$

But $L = V(x,u,t) + \frac{d\varphi}{dt} + \lambda(t)(f(x,u,t) - \dot{x})$

$$L = H(x,u,\lambda,t) + \frac{d\varphi}{dt} - \lambda(t)\dot{x}(t) \tag{18}$$

$$\frac{\partial l}{\partial x} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right) = \frac{\partial H}{\partial x} + \frac{\partial}{\partial x}\left(\frac{d\varphi}{dt}\right) - 0 - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{d\varphi}{dt}\right) - \lambda(t)\right) \tag{19}$$

Note that

$$\frac{d\varphi}{dt} = \frac{d\varphi(x(t),t)}{dt} = \frac{\partial\varphi(x,t)}{\partial x}\dot{x} + \frac{\partial\varphi(x,t)}{\partial t} \tag{20}$$

We have

$$\frac{\partial l}{\partial x} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right) = \frac{\partial H}{\partial x} + \frac{\partial^2\varphi(x,t)}{\partial x^2}\dot{x} + \frac{\partial^2\varphi(x,t)}{\partial x\,\partial t} - \frac{d}{dt}\left(\frac{\partial\varphi(x,t)}{\partial x} - \lambda(t)\right)$$
$$= \frac{\partial H}{\partial x} + \frac{\partial^2\varphi(x,t)}{\partial x^2}\dot{x} + \frac{\partial^2\varphi(x,t)}{\partial x\,\partial t} - \left[\frac{\partial^2\varphi(x,t)}{\partial x^2}\dot{x} + \frac{\partial^2\varphi(x,t)}{\partial x\partial t} - \frac{d\lambda}{dt}\right] = \frac{\partial H}{\partial x} + \frac{d\lambda}{dt} = 0 \tag{21}$$

From (18)

$$\frac{\partial l}{\partial u} = \frac{\partial H}{\partial u} = 0 \tag{22}$$
$$\frac{\partial l}{\partial \lambda} = \frac{\partial H}{\partial \lambda} - \dot{x}(t) = 0 \tag{23}$$

Step 4:

The necessary conditions are

$$\frac{\partial l}{\partial x} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right) = 0 \Longrightarrow \frac{\partial H}{\partial x} = -\dot{\lambda}(t) \tag{24}$$
$$\frac{\partial l}{\partial u} = 0 \Longrightarrow \frac{\partial H}{\partial u} = 0 \tag{25}$$

$$L(x,\dot{x},u,\lambda)\Big|_{t=t_f}\Delta t_f+\frac{\partial l}{\partial \dot{x}}\Delta x\Big|_{t_0=0}^{t_f} \tag{26}$$

$$\Rightarrow\left[H+\frac{\partial\varphi}{\partial t}\right]\Big|_{t=t_f}\Delta t_f+\left[\frac{\partial\varphi}{\partial x}-\lambda(t)\right]\Big|_{t=t_f}\Delta x_f=0 \tag{27}$$

$$\frac{\partial l}{\partial\lambda}=0\Rightarrow\frac{\partial H}{\partial\lambda}=\dot{x}(t) \tag{28}$$

Example

Find the extrema (i.e the optimal control and optimal state)

$$J(u)=\frac{1}{2}\int_{t_0}^{t_f}u^2(t)dt$$

Under the conditions $\dot{x}_1(t)=x_2(t)\quad,\quad t_0=0$

$\dot{x}_2(t)=u(t)\quad,\quad t_f=2$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}(t_0)=\begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}(t_f)=\begin{bmatrix}1\\0\end{bmatrix}$$

Solution

From the problem, we identify the following

$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}\quad,\qquad V(x,u,t)=V(u(t))=\frac{1}{2}u^2(t)$

$f(x,u,t)=\begin{bmatrix}f_1\\f_2\end{bmatrix}$ Where $f_1=x_2\ ,\ f_2=u$

Step 1:

We formulate the Hamiltonian function as

$H=H(x_1(t),x_2(t),u(t),\lambda_1(t),\lambda_2(t))=V+\lambda f$

$=\frac{1}{2}u^2(t)+(\lambda_1,\lambda_2)\begin{bmatrix}x_2\\u\end{bmatrix}$

$=\frac{1}{2}u^2(t)+\lambda_1x_2+\lambda_2u$

Step2:

Find u from

$$\frac{\partial H}{\partial u}=0\Rightarrow u+\lambda_2=0\Rightarrow u=-\lambda_2$$

Step3:

Using the results of step2 in step1, find the optimal

$H=\frac{1}{2}\lambda_2^2+\lambda_1x_2-\lambda_2^2$

$=\lambda_1x_2-\frac{1}{2}\lambda_2^2$

Step4:

Obtain the state and costate equations from

$H(x_1,x_2,\lambda_1,\lambda_2)=\lambda_1x_2-\frac{1}{2}\lambda_2^2$

$\frac{\partial H}{\partial x_1}=0\quad,\quad\frac{\partial H}{\partial x_2}=\lambda_1$

$\frac{\partial H}{\partial\lambda_1}=x_2\quad,\quad\frac{\partial H}{\partial\lambda_2}=-\lambda_2$

$$\begin{bmatrix}\frac{\partial H}{\partial x_1}\\\frac{\partial H}{\partial x_2}\end{bmatrix}+\frac{d}{dt}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}=0$$

$$\begin{bmatrix}0\\\lambda_1\end{bmatrix}+\begin{bmatrix}\frac{d\lambda_1}{dt}\\\frac{d\lambda_2}{dt}\end{bmatrix}=0\Rightarrow\frac{d\lambda_1}{dt}=0\rightarrow\lambda_1=c_1$$

$\lambda_1 + \frac{d\lambda_2}{dt} = 0 \to c_1 + \frac{d\lambda_2}{dt} = 0 \Rightarrow \lambda_2 = -c_1 t + c_2$

$\frac{\partial H}{\partial \lambda_2} = -\lambda_2 = u$

$\dot{x}_2 = \lambda_2 \Rightarrow \dot{x}_2 = -(-c_1 t + c_2) = c_1 t - c_2$

$\frac{dx_2}{dt} = c_1 t - c_2$

$x_2 = \frac{c_1 t^2}{2} - c_2 t + c_3$

$\dot{x}_1 = x_2$

$\dot{x}_1 = \frac{c_1 t^2}{2} - c_2 t + c_3 \Rightarrow x_1 = \frac{c_1 t^3}{6} - \frac{c_2 t^2}{2} + c_3 t + c_4$

Step5:

Obtain the optimal control from

$u = -\lambda_2$

$u = c_1 t - c_2$

Where c_1, c_2, c_3 and c_4 are constants evaluated using the boundary conditons given that is

$$\begin{bmatrix} \frac{c_1 t^3}{6} - \frac{c_2 t^2}{2} + c_3 t + c_4 \\ \frac{c_{1t^2}}{2} - c_2 t + c_3 \end{bmatrix} (t_0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} c_4 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow c_4 = 1, c_3 = 2$$

Also

$$\begin{bmatrix} \frac{c_1 t^3}{6} - \frac{c_2 t^2}{2} + c_3 t + c_4 \\ \frac{c_{1t^2}}{2} - c_2 t + c_3 \end{bmatrix} (t_f) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \frac{8c_1 t^3}{6} - \frac{4c_2 t^2}{2} + 2c_3 t + c_4 \\ \frac{4c_1}{2} - 2c_1 + c_3 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solving these equations simultaneously gives $c_2 = 4$, $c_1 = 3$

Finally, we have the optimal states constates and control as

$x_1(t) = 0.5t^3 - 2t^2 + 2t + 1$

$x_2(t) = 1.5t^2 - 4t + 2$

$\lambda_1(t) = 3$

$\lambda_2(t) = -3t^3 + 4$

$u(t) = 3t - 4$

$J(u) = \frac{1}{2}\int_0^2 (3t-4)^2 dt = \frac{1}{2}[24 + 32 - 48] = 4$

3. Linear Quadratic

In this section we explore exactly this approach for the linear quadratic optimal control problem.

Consider the optimal control for arbitrary for the system

$$\dot{x} = ax + bu \qquad (29)$$

Where $x = R$ is a scalar state , $u \epsilon R$ is the input, the initial state $x(t_0)$ is given and $a, b \epsilon R$ are positive constants.

We wish to find a trajectory $(x(t), u(t))$ that minimizes the cost function.

$$J(u) = \tfrac{1}{2}\int_{t_0}^{t_f} u^2(t)dt + \tfrac{1}{2}cx^2(t_f) \qquad (30)$$

Where the terminal time t_f is given and $c > 0$ is a constant. This cost function balances the final value of the state with the input required to get to that state.

To solve the problem, we define:

$$V = \tfrac{1}{2}u^2(t) \qquad (31)$$

$$\varphi = \frac{1}{2} c x^2(t_f) \tag{32}$$

We write the Hamiltonian of this system and derive the following expressions for the costate λ:

$$H = V + \lambda f = \frac{1}{2} u^2 + \lambda(ax + bu) \tag{33}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -a\lambda \ , \ \lambda(t_f) = \frac{\partial \varphi}{\partial x} = cx(t_f) \tag{34}$$

This ia a final value problem for a linear differential equation in λ and the solution can be shown to be

$$\lambda(t) = cx(t_f)e^{a(t_f - t)} \tag{35}$$

The optimal control is given by

$$\frac{\partial H}{\partial u} = u + b\lambda = 0 \Longrightarrow u^*(t) - b\lambda(t) = -bcx(t_f)e^{a(t_f - t)} \tag{36}$$

Substituting this control into the dynamics given by equation (29) yields a first-order ODE in x:

$$\dot{x} = ax - b^2 cx(t_f)e^{a(t_f - t)} \tag{37}$$

This can be solved explictiy as

$$x^*(t) = x(t_0)e^{a^{(t-t_0)}} + \frac{b^2 c}{2a} x^*(t_f)[e^{a(t_f - t)} - e^{a(t + t_f - 2t_0)}] \tag{38}$$

Setting $t = t_f$ and solving for $x(t_f)$,Gives

$$x^*(t_f) = \frac{2a \, e^{a(t_f - t_0)} x(t_0)}{2a - b^2 c(1 - e^{2a(t_f - t_0)})} \tag{39}$$

And finally we can write

$$u^*(t) = -\frac{2abc \ e^{a(2t_f - t_0 - t)} x(t_0)}{2a - b^2 c(1 - e^{2a(t_f - t_0)})} \tag{40}$$

$$x^*(t) = x(t_0)e^{a(t - t_0)} + \frac{b^2 c \ e^{a(t_f - t_0)} x(t_0)}{2a - b^2 c\left(1 - e^{2a(t_f - t_0)}\right)} [e^{a(t_f - t)} - e^{a(t + t_f - 2t_0)} \tag{41}$$

We can use form of this expression to explore how our cost function effect the optimal trajectory.
For example, we can ask what happens to the terminal state $x^*(t_f)$ and $c \to \infty$.
Setting $t = t_f$ in equation (41) and taking the limit we find that

$$\lim_{c \to \infty} x^*(t_f) = 0 \tag{42}$$

4. Conclusions

In this work we can use expression to explore how our cost function effects the optimal trajectory, also using these tools we derive the linear quadratic regulator for linear systems and describe its usage.

Acknowledgements

Author of contributors to this paper helpful comments thanks.

References

[1] Anderson,B.D.O. and J.B. Moore, Optimal Control, Linear Quadralic Methods, prentice-hall, Englewood C liffs, NJ, 1989.
[2] M.E.J. Newman. M. Girvan, and J.D. Farmer, Optimal Design. Robustness and Risk Aversion, phys. ReV. Lett.,89,028301(2002)
[3] Habib.H.Molaei, Gradient in Optimal Control Problem With Non-local Boundary Conditions. Transactions of NASA, Series of Mathematical science and technical, Baku, XXVI, No 7 (2006).
[4] Molaei H.H. Difference Approximation of Quadrature Optimal Control Problem With Integral Conditions. The 1-st International with Industrial Applications, Baku, Azerbaijan, May 22-25,2006, pp73-74.
[5] Bryson, A.E.and Ho,Y.C.,Applied Optimal Control, Hemisphere Publishing Company, 1975.