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Class One and Class Two Graphs

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Abstract:

This paper studies class one and class two graphs. The main results are 1) Every regular graph of odd order is class two. 2) For a non empty bipartite graph, chromatic index is equal to maximum degree of graph.

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Key words: bipartite graph, complete graphs, cycles, chromatic index, degree of graph.

§ 0 Introduction.

Basically, a major question in the area of edge colorings is that of determining to which of these two classes a given graph belongs. While we will see many graphs of Class one and many graphs of Class two, it turns out that it is much more likely that a graph is of Class one. Paul Erdos and Robin J. Wilson **[1]** proved the following, where the set of graphs of order n is denoted by G_n and the set of graphs of order n and of Class one is denoted by $G_{n,1}$.

§ 1 .Class one and Class two graphs

1.1. Definition.

A graph G belongs to Class one if $\chi'(G) = \Delta(G)$ and is of Class two if $\chi'(G) = 1 + \Delta(G)$.

1.2. Theorem. Almost every graph is of class one, that is,

$$\lim_{n \to \infty} \frac{|G_{n,1}|}{|G_n|} = 1$$

Proof: We now look at a few well-known graph and classes of graphs to determine whether they are of Class one or Class two. We begin with the cycles. Since the cycle $C_n (n \geq 3)$ is 2-regular, $\chi'(C_n) = 2$ or $\chi'(C_n) = 3$. If n is even, then the edges may be alternately colored 1 and 2, producing a 2-edge coloring of C_n. If n is odd, then $\alpha'(C_n) = (n-1)/2$. Since the size of C_n is n, it follows by that $\chi'(C_n) \geq n / \alpha'(C_n) = 2n/(n-1) > 2$ and so $\chi'(C_n) = 3$.

Therefore,

$$\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Since $\Delta(C_n) = 2$, it follows that C_n is of Class one if n is even and of Class two if n is odd.

We now turn to complete graphs. Since K_n is $(n-1)$-regular, either $\chi'(K_n) = n-1$ or $\chi'(K_n) = n$. If n is even, then it follows that K_n is 1-factorable, that is, K_n can be factored into $n-1$ 1-factors $F_1, F_2, ..., F_{n-1}$. By assigning each edge of F_i $(1 \leq I \leq n-1)$ the color i, an $(n-1)$-edge coloring of K_n is produced. If n is odd, then $\alpha'(K_n) = (n-1)/2$. Since the size m of K_n is $n(n-1)/2$, it follows that $\chi'(K_n) \geq m/\alpha'(K_n) = n$. Thus $\chi'(K_n) = n$. In summary,

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the chromatic index of every nonempty complete graph is an odd integer **[3].** Since $\Delta(K_n) = n-1$, it follows, as with the cycles C_n, that K_n is of Class one if n is even and of Class two if n is odd. Of course, both the cycles and complete graphs are regular graphs. For an r-regular graph G, either $\chi'(G) = r$ or $\chi'(G) = r + 1$. If $\chi'(G) = r$, then there is an r-edge coloring of G, resulting in r color classes $E_1, E_2, ..., E_r$. Since every vertex v of G has degree r, the vertex v is incident with exactly one edge in each set E_i $(1 \leq i \leq r)$. Therefore, each color class E_i is a perfect matching and G is 1-factorable. Conversely, if G is 1-factorable, then $\chi'(G) = r$.

1.3 Theorem. A regular graph G is of Class one if and only if G is 1-factorable.

Proof: The Petersen graph P is not 1-factorable and so it is of Class two, that is, $\chi'(P) = 4$. The formulas mentioned above for the chromatic index of cycles and complete graphs are immediate consequences of Theorem 1.3, as is the following.

1.4. Corollary. Every regular graph of odd order is of Class two. We have already seen that the even cycles are of Class one. The four graphs shown in **Figure 1** are of Class one as well. The even cycles and the four graphs of **Figure 1** are all bipartite. These graphs serve as illustrations of a theorem due to Denes Konig **[4, 5].**

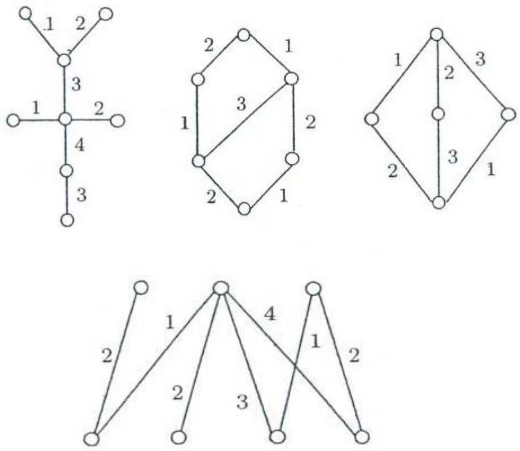

Figure1: some class one graph

1.5. Theorem. If G is a nonempty bipartite graph, then

$$\chi'(G) = \Delta(G)$$

Proof. Suppose that the theorem is false. Then among the counterexamples, let G be one of minimum size. Thus G is a bipartite graph such that $\chi'(G) = \Delta(G) + 1$. Let $e \in E(G)$, where $e = uv$. Necessarily, u and v belong to different partite sets of G. Then $\chi'(G\text{-}e) = \Delta(G\text{-}e)$. Now $\Delta(G\text{-}e) = \Delta(G)$, for otherwise G is $\Delta(G)$-edge colorable.

Let there be given a $\Delta(G)$-edge coloring of $G\text{-}e$. Each of the $\Delta(G)$ colors must be assigned to an edge incident either with u or with v in $G\text{-}e$, for otherwise this color could be assigned to e producing a $\Delta(G)$-edge coloring of G. Because $deg_{G\text{-}e}u < \Delta(G)$ and $deg_{G\text{-}e}v < \Delta(G)$, there is a color α of the $\Delta(G)$ colors not used in coloring the edges of $G\text{-}e$ incident with u and a color β of the $\Delta(G)$ colors not used in coloring the edges of $G\text{-}e$ incident with v . Then $\alpha \neq \beta$ and, furthermore, some edge incident with v is colored α and some edge incident with u is colored β **[2].**

Let P be a path of maximum length having initial vertex v whose edges are alternately colored α and β . The path P cannot contain u, for otherwise P has odd length, implying that the initial and terminal edges of P are both colored α. This is impossible, however, since u is incident with no edge colored α . Interchanging the colors α and β of the edges of P produces a new $\Delta(G)$-edge coloring of $G\text{-}e$ in which neither u nor v is incident with an edge colored α. Assigning e the color α produces a $\Delta(G)$-edge coloring of G, which is a contradiction.

We have seen that if G is a graph of size m, then any partition of $E(G)$ into independent sets must contain at least $\frac{m}{\alpha'(G)}$ sets. If v is a vertex with $deg\ v = \Delta(G)$, then each of the $\Delta(G)$ edges incident with v must belong to distinct independent sets. Thus $\frac{m}{\alpha'(G)} \geq \Delta(G)$ and so $m \geq \Delta(G).\ \alpha'(G)$. If $m > \Delta(G).\alpha'(G)$, then we can say more.

1.6. Theorem. If G is a graph of size m such that

$$m > \alpha'(G)\ \Delta(G),$$

then G is of class two.

Proof. We know $\chi'(G) \geq \frac{m}{\alpha'(G)}$. Thus

$$\chi'(G) \geq \frac{m}{\alpha'(G)} > \frac{\Delta(G).\alpha'(G)}{\alpha'(G)} = \Delta(G),$$

which implies that $\chi(G) = 1 + \Delta(G)$ and so G is of Class two.

1.7. Example. A community, well known for having several professional tennis players train there, holds a charity tennis tournament each year, which alternates between men and women tennis players. During the coming year, women tennis players will be featured and the professional players Alice, Barbara, and Carrie will be in charge. Two tennis players from each of two local tennis clubs have been invited to participate as well. Debbie and Elizabeth will participate from Woodland Hills Tennis Club and Frances and Gina will participate from Mountain Meadows Tennis Club. No two professionals will play each other in the tournament and no two players from the same tennis club will play each other; otherwise, every two of the seven players will play each other. If no player is to play two matches on the same day, what is the minimum number of days needed to schedule this tournament?

Solution. We construct a graph H with $V(H) = \{A, B\ldots G\}$ whose vertices correspond to the seven tennis players. Two vertices x and y are adjacent in H if x and y are to play a tennis match against each other. The graph H is shown in **figure 2**. The answer to the question posed is the chromatic index of H. The order of H is $n=7$ and the degrees of its vertices are 5,5,5,5,4,4,4. Thus $\Delta(H) = 5$ and the size of H is $m = 16$. Since

$$16 = m > \Delta(H).\left\lfloor \frac{n}{2} \right\rfloor = 15,$$

The graph H is overfull. Since every overfull graph

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is of class two, H is of Class two and so $\chi'(H) = 1 + \Delta(H) = 6$. A 6-edge coloring of H is also shown in **Figure 2**. This provides us with a schedule for the tennis tournament taking place over a minimum of six days.

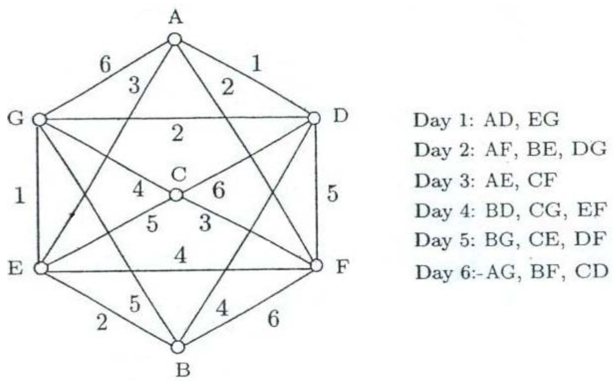

Figure 2: The graph *H* in Example 1.7 and a 6-edge coloring of *H*

1.8. Example. One year it is decided to have a charity tennis tournament consisting entirely of double matches. Five tennis players (denoted by A, B, C, D, E) have agreed to participate. Each pair $\{W, X\}$ of tennis players will play a match against every other pair $\{Y, Z\}$ of tennis players, where then $\{W, X\} \cap \{Y, Z\} = \phi$, but no 2-person team is to play two matches on the same day. What is the minimum number of days needed to schedule such a tournament? Give an example of such a tournament using a minimum number of days.

Solution. We construct a graph G whose vertex set is the set of 2-element subsets of $\{A, B, C, D, \text{and } E\}$. Thus the order of G is $\binom{5}{2} = 10$. Two vertices $\{W, X\}$ and $\{Y, Z\}$ are adjacent if these sets are disjoint. The graph G is shown in **Figure 3**. Thus G is the Petersen graph, or equivalently the Kneser graph $KG_{5,2}$. To answer the question, we determine the chromatic index of G. Since the Petersen graph is known to be of Class two, it follows that $\chi'(G) = 1 + \Delta(G) = 4$. A 4-edge coloring of G is given in **Figure 3** together with a possible schedule of tennis matches over a period of four days.

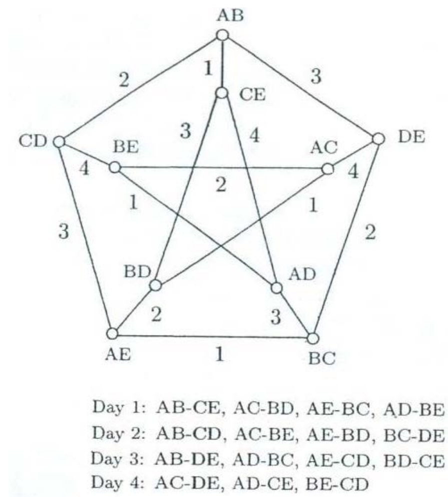

Figure 3: the Petersen graph *G* in Example and a 4 edge coloring of *G*

As we have seen, the size of a graph G of Class one and having order n cannot exceed $\Delta(G).\left\lfloor \frac{n}{2} \right\rfloor$. The size of any overfull graph of order n exceeds this number and is therefore of Class two. There are related sub graphs such that if a graph G should contain one of these, then G must also be of Class two.

A subgraph H of odd order n' and size m' of a graph G is an overfull subgraph of G if

$$m' > \Delta(G).\left\lfloor \frac{n'}{2} \right\rfloor = \Delta(G).\left\lfloor \frac{n'-1}{2} \right\rfloor$$

Actually, if H is an overfull subgraph of G, then $\Delta(H) = \Delta(G)$. This says that H is itself of Class two. Not only is an overfull subgraph of a graph G of Class two, G itself is of Class two.

1.9 Remark:

1) All regular graphs of even order, bipartite graphs are class one.
2) Petersen graph, cycles and complete graphs are class two.

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