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The Number of Zeros of a Polynomial in a Disk

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Abstract: In this paper we subject the coefficients of a polynomial and their real and imaginary parts to certain restrictions and give bounds for the number of zeros in a specific region. Our results generalize many previously known results and imply a number of new results as well.

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1. Introduction

A large number of research papers have been published so far on the location in the complex plane of some or all of the zeros of a polynomial in terms of the coefficients of the polynomial or their real and imaginary parts. The famous Enestrom-Kakeya Theorem states [6] that if the coefficients of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ satisfy

$$0 \le a_0 \le a_1 \le \ldots\ldots \le a_{n-1} \le a_n ,$$

then all the zeros of P(z) lie in the closed disk $|z| \le 1$. By putting a restriction on the coefficients of a polynomial similar to that of the Enestrom-Kakeya Theorem, Mohammad [7] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial such that

$$0 < a_0 \le a_1 \le \ldots\ldots \le a_{n-1} \le a_n .$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

For polynomials with complex coefficients, Dewan [1] proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial such that

$$\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,2,.....,n,$$

for some real α and β and

$$0<\left|a_0\right|\le\left|a_1\right|\le......\le\left|a_{n-1}\right|\le\left|a_n\right|.$$

Then the number of zeros of P(z) in $|z|\le\frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{\left|a_n\right|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}\left|a_j\right|}{\left|a_0\right|}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that

$$0<\alpha_0\le\alpha_1\le......\le\alpha_{n-1}\le\alpha_n.$$

Then the number of zeros of P(z) in $|z|\le\frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_0\right|}.$$

Regarding the number of zeros of P(z) in $|z|\le\delta, 0<\delta<1$, Gulzar [4,5] proved the following generalizations of the above results:

Theorem D: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that for some $0<k\le 1, 0<\tau\le 1$ and $0\le l\le n$,

$$k\alpha_n\le\alpha_{n-1}\le.......\le\alpha_{l+1}\le\alpha_l\ge\alpha_{l-1}\ge......\ge\alpha_1\ge\tau\alpha_0.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le\delta$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|},$$

where

$$M = 2\alpha_l + k(|\alpha_n| - \alpha_n) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|.$$

Theorem E: : Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that for some $0<k_1,k_2\le 1, 0<\tau_1,\tau_2\le 1$ **,** $0\le l\le n$ and $0\le m\le n$,

$$k_1\alpha_n \le \alpha_{n-1} \le .... \le \alpha_{l+1} \le \alpha_l \ge \alpha_{l-1} \ge ...... \ge \alpha_1 \ge \tau_1\alpha_0$$

$$k_2\beta_n \le \beta_{n-1} \le .... \le \beta_{m+1} \le \beta_m \ge \beta_{m-1} \ge ...... \ge \beta_1 \ge \tau_2\beta_0.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le\delta$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M'}{|a_0|},$$

where

$$M' = |a_n| + (1-k_1)|\alpha_n| + (1-k_2)|\beta_n| + 2(\alpha_l + \beta_m) - (k_1\alpha_n + k_2\beta_n) + (1-\tau_1)|\alpha_0|$$

$$-\tau_1\alpha_0 + (1-\tau_2)|\beta_0| - \tau_2\beta_0.$$

Recently, Gardner and Sheilds [2] proved the following results:

Theorem F: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some t>0, $0\le l\le n$ and for some real numbers α,β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,.....,n,$$

and

$$t^n|a_n| \le t^{n-1}|a_{n-1}| \le .... \le t^{l+1}|a_{l+1}| \le t^l|a_l| \ge t^{l-1}|a_{l-1}| \ge ...... \ge t|a_1| \ge |a_0| > 0.$$

Then the number of zeros of P(z) in $|z|\le\delta t, 0<\delta<1$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M^*}{|a_0|}$, where

$$M^* = |a_0|t(1-\cos\alpha-\sin\alpha) + 2|a_l|t^{l+1}\cos\alpha + |a_n|t^{n+1}(1+\sin\alpha-\cos\alpha) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|t^{j+1}.$$

Theorem G: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that for some t>0, and some $0\le l\le n$,

$$t^n\alpha_n \le t^{n-1}\alpha_{n-1}\le ......\le t^{l+1}\alpha_{l+1}\le t^l\alpha_l \ge t^{l-1}\alpha_{l-1}\ge ......\ge t\alpha_1 \ge \alpha_0 \ne 0.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le \delta t$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M^{**}}{|a_0|}$,

where

$$M^{**}=(|\alpha_0|-\alpha_0)t+2\alpha_l t^{l+1}+(|\alpha_n|-\alpha_n)t^{n+1}+2\sum_{j=0}^{n}|\beta_j|t^{j+1}.$$

Theorem H: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that for some t>0, **,** $0\le l\le n$ and $0\le m\le n$,

$$t^n\alpha_n \le t^{n-1}\alpha_{n-1}\le ......\le t^{l+1}\alpha_{l+1}\le t^l\alpha_l \ge t^{l-1}\alpha_{l-1}\ge ......\ge t\alpha_1 \ge \alpha_0 \ne 0$$

$$t^n\beta_n \le t^{n-1}\beta_{n-1}\le ......\le t^{m+1}\beta_{l+1}\le t^m\beta_m \ge t^{m-1}\beta_{m-1}\ge ......\ge t\beta_1 \ge \beta_0.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le \delta t$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M^{***}}{|a_0|}$,

where

$$M^{***}=(|\alpha_0|-\alpha_0)t+2\alpha_l t^{l+1}+_1(|\alpha_n|-\alpha_n)t^{n+1}+(|\beta_0|-\beta_0)t+2\beta_m t^{m+1}+(|\beta_n|-\beta_n)t^{n+1}.$$

In this paper, we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some t>0, $0\le l\le n$, $0<k\le 1, 0<\tau\le 1$ and for some real numbers α,β,

$$|\arg a_j-\beta|\le \alpha\le \frac{\pi}{2},\ j=0,1,2,.....,n,$$

and

$$kt^n|a_n|\le t^{n-1}|a_{n-1}|\le ......\le t^{l+1}|a_{l+1}|\le t^l|a_l|\ge t^{l-1}|a_{l-1}|\ge ......\ge t|a_1|\ge \tau|a_0|>0.$$

Then the number of zeros of P(z) in $|z|\le \delta t, 0<\delta<1$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M_1}{|a_0|}$, where

$$M_1=2|a_0|t-\tau|a_0|(1+\cos\alpha-\sin\alpha)+2|a_l|t^{l+1}\cos\alpha+|a_n|t^{n+1}(1+k\sin\alpha-k\cos\alpha)$$

$$+(1-k)|a_n|t^n + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|t^{j+1}.$$

Remark 1: For different values of the parameters, we get many interesting results. For example for k=1, τ =1, Theorem 1 reduces to Theorem F . For t=1, it reduces to Theorem D .

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that for some t>0, $0<k\le 1, 0<\tau\le 1$ and $0\le l\le n$,

$$kt^n\alpha_n \le t^{n-1}\alpha_{n-1}\le ....\le t^{l+1}\alpha_{l+1}\le t^l\alpha_l \ge t^{l-1}\alpha_{l-1}\ge ......\ge t\alpha_1 \ge \tau\alpha_0 \ne 0.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le\delta t$ is less than $\dfrac{1}{\log\frac{1}{\delta}}\log\dfrac{M_2}{|a_0|}$,

where

$$M_2 = 2|\alpha_0|t-\tau(|\alpha_0|+\alpha_0)t+2\alpha_l t^{l+1}+2|\alpha_n|t^{n+1}-k(|\alpha_n|+\alpha_n)t^{n+1}+2\sum_{j=0}^{n}|\beta_j|t^{j+1}.$$

Remark 2: For different values of the parameters, we get many interesting results. For example for k=1, τ =1, Theorem 2 reduces to Theorem G . For t=1, it reduces to the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j,\ j=0,1,2,......,n$ such that for some $0<k\le 1, 0<\tau\le 1$ and $0\le l\le n$,

$$k\alpha_n \le \alpha_{n-1}\le ....\le \alpha_{l+1}\le \alpha_l \ge \alpha_{l-1}\ge ......\ge \alpha_1 \ge \tau\alpha_0 \ne 0.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le\delta$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M_2^*}{|a_0|},$$

where

$$M_2^* = 2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\alpha_l+2|\alpha_n|-k(|\alpha_n|+\alpha_n)+2\sum_{j=0}^{n}|\beta_j|.$$

Applying Theorem 2 to the polynomial –iP(z) , we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$,

$\text{Im}(a_j) = \beta_j, j = 0,1,2,......,n$ such that for some t>0, $0 < k \le 1, 0 < \tau \le 1$ and $0 \le l \le n$,

$$kt^n \beta_n \le t^{n-1}\beta_{n-1} \le .... \le t^{l+1}\beta_{l+1} \le t^l \beta_l \ge t^{l-1}\beta_{l-1} \ge ...... \ge t\beta_1 \ge \tau\beta_0.$$

Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta t$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M_2^{**}}{|a_0|}$,

where

$$M_2^{**} = 2|\beta_0|t - \tau(|\beta_0| + \beta_0)t + 2\beta_l t^{l+1} + 2|\beta_n|t^{n+1} - k(|\beta_n| + \beta_n)t^{n+1} + 2\sum_{j=0}^{n}|\alpha_j|t^{j+1}.$$

If a_j is real i.e. $\beta_j = 0, \forall j$, Theorem 2 gives the following result:

Corollary 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ b e a polynomial of degree n such that for some t>0, $0 < k \le 1, 0 < \tau \le 1$ and $0 \le l \le n$,

$$kt^n a_n \le t^{n-1}a_{n-1} \le .... \le t^{l+1}a_{l+1} \le t^l a_l \ge t^{l-1}a_{l-1} \ge ...... \ge ta_1 \ge \tau a_0 \ne 0.$$

Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta t$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M_2^{***}}{|a_0|}$,

where

$$M_2^{***} = 2|a_0|t - \tau(|a_0| + a_0)t + 2a_l t^{l+1} + 2|a_n|t^{n+1} - k(|a_n| + a_n)t^{n+1}.$$

Theorem 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ b e a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$, $\text{Im}(a_j) = \beta_j, j = 0,1,2,......,n$ such that for some t>0, $0 < k_1, k_2 \le 1, 0 < \tau_1, \tau_2 \le 1$ **,** $0 \le l \le n$ and $0 \le m \le n$,

$$k_1 t^n \alpha_n \le t^{n-1}\alpha_{n-1} \le .... \le t^{l+1}\alpha_{l+1} \le t^l \alpha_l \ge t^{l-1}\alpha_{l-1} \ge ...... \ge t\alpha_1 \ge \tau_1\alpha_0 \ne 0$$

$$k_2 t^n \beta_n \le t^{n-1}\beta_{n-1} \le .... \le t^{l+1}\beta_{l+1} \le t^l \beta_m \ge t^{l-1}\beta_{m-1} \ge ...... \ge t\beta_1 \ge \tau_2\beta_0.$$

Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta t$ is less than $\frac{1}{\log\frac{1}{\delta}}\log\frac{M_3}{|a_0|}$,

where

$$M_3 = 2|\alpha_0|t - \tau_1(|\alpha_0| + \alpha_0)t + 2\alpha_l t^{l+1} + |\alpha_n|t^{n+1} - k_1(|\alpha_n| + \alpha_n)t^{n+1}$$

$$+ 2|\beta_0|t - \tau_1(|\beta_0| + \beta_0)t + 2\beta_m t^{m+1} + |\beta_n|t^{n+1} - k_2(|\beta_n| + \beta_n)t^{n+1}.$$

Remark 3: For different values of the parameters, we get many interesting results. For example for k=1, τ =1, Theorem 3 reduces to Theorem H . For t=1, it reduces to Theorem E .

2. Lemmas

For the proofs of the above results, we need the following results:

Lemma 1: Let $z_1, z_2 \in C$ with $|z_1| \geq |z_2|$ and $|\arg z_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1,2$ for some real numbers α and β. Then

$$|z_1 - z_2| \leq (|z_1| - |z_2|)\cos\alpha + (|z_1| + |z_2|)\sin\alpha .$$

The above lemma is due to Govil and Rahman [3].

Lemma 2: Let F(z) be analytic in $|z| \leq R$, $|F(z)| \leq M$ for $|z| \leq R$ and $F(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of F(z) in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}.$$

For the proof of this lemma see [8, P.171].

3. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$F(z) = (t-z)P(z) = (t-z)(a_0 + a_1 z) + a_2 z^2 + ... + a_{l-1}z^{l-1} + a_l z^l + a_{l+1}z^{l+1} + ...a_{n-1}z^{n-1} + a_n z^n$$

$$= a_0 t + (a_1 t - a_0)z + ... + (a_l t - a_{l-1})z^l + (a_{l+1}t - a_l)z^{l+1} + ... + (a_n t - a_{n-1})z^n - a_n z^{n+1}$$

$$= a_0 t + [(a_1 t - \tau a_0) + (\tau a_0 - a_0)]z + (a_2 t - a_1)z^2 + ... + (a_l t - a_{l-1})z^l + (a_{l+1}t - a_l)z^{l+1}$$

$$+ ...... + [(ka_n t - a_{n-1}) - (ka_n t - a_n t)]z^n - a_n z^{n+1}$$

For $|z| = t$, we have by using the hypothesis and Lemma 1,

$$|F(z)| \leq |a_0|t + |\tau a_0 - a_0|t + |a_1 t - \tau a_0|t + |a_2 t - a_1|t^2 + ... + |a_l t - a_{l-1}|t^l + |a_{l+1}t - a_l|t^{l+1}$$

$$+ ...... + |ka_n t - a_{n-1}|t^n + |ka_n t - a_n|t^n + |a_n|t^{n+1}$$

$$= |a_0|t + (1-\tau)|a_0|t + |a_1 t - a_0|t + |a_2 t - a_1|t^2 + ... + |a_l t - a_{l-1}|t^l + |a_{l+1}t - a_l|t^{l+1}$$

$$+ ...... + |ka_n t - a_{n-1}|t^n + (1-k)|a_n|t^n + |a_n|t^{n+1}$$

$$\le |a_0|t + (1-\tau)|a_0|t + [(|a_1|t - \tau|a_0|)\cos\alpha + (|a_1|t + \tau|a_0|)\sin\alpha]t$$

$$+[(|a_2|t - |a_1|)\cos\alpha + (|a_2|t + |a_1|)\sin\alpha]t^2 + ...$$
$$+[(|a_l|t - |a_{l-1}|)\cos\alpha + (|a_l|t - |a_{l-1}|)\sin\alpha]t^l$$
$$+[(|a_l| - |a_{l+1}|t)\cos\alpha + |a_l| + |a_{l+1}|t)\sin\alpha]t^{l+1} + ...$$
$$+[(|a_{n-1}| - k|a_n|t)\cos\alpha + (|a_{n-1}| + k|a_n|t)\sin\alpha]t^n + (1-k)|a_n|t^n + |a_n|t^{n+1}$$

$$= 2|a_0|t - \tau|a_0|(1+\cos\alpha - \sin\alpha) + 2|a_l|t^{l+1}\cos\alpha + |a_n|t^{n+1}(1 + k\sin\alpha - k\cos\alpha)$$
$$+(1-k)|a_n|t^n + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|t^{j+1}$$
$$= M_1.$$

Since F(z) is analytic for $|z| \le t$ and $|F(z)| \le M_1$ for $|z| = t$, it follows by Lemma 2 and Maximum Modulus Theorem that the number of zeros of F(z) and hence of P(z) in $|z| \le \delta t$ is less than or equal to $\dfrac{1}{\log\frac{1}{\delta}}\log\dfrac{M_1}{|a_0|}$ and the theorem follows.

Proof of Theorem 2: Consider the polynomial

$$F(z) = (t-z)P(z) = (t-z)(a_0 + a_1 z) + a_2 z^2 + ... + a_{l-1}z^{l-1} + a_l z^l + a_{l+1}z^{l+1} + ...a_{n-1}z^{n-1} + a_n z^n$$
$$= a_0 t + (a_1 t - a_0)z + ... + (a_l t - a_{l-1})z^l + (a_{l+1}t - a_l)z^{l+1} + ... + (a_n t - a_{n-1})z^n - a_n z^{n+1}$$
$$= a_0 t + [(a_1 t - \tau a_0) + (\tau a_0 - a_0)]z + (a_2 t - a_1)z^2 + ... + (a_l t - a_{l-1})z^l + (a_{l+1}t - a_l)z^{l+1}$$

$$+ ...... + [(ka_n t - a_{n-1}) - (ka_n t - a_n t)]z^n - a_n z^{n+1}$$
$$= (\alpha_0 + i\beta_0)t + [\{(\alpha_1 + i\beta_1)t - \tau(\alpha_0 + i\beta_0)\} + \{\tau(\alpha_0 + i\beta_0) - (\alpha_0 + i\beta_0)\}]z$$
$$+ ...... + [(\alpha_l + i\beta_l)t - (\alpha_{l-1} + i\beta_{l-1})]z^l + [(\alpha_{l+1} + i\beta_{l+1})t - (\alpha_l + i\beta_l)]z^{l+1}$$
$$+ ...... + [\{k(\alpha_n + i\beta_n)t - (\alpha_{n-1} + i\beta_{n-1})\} - \{k(\alpha_n + i\beta_n)t - (\alpha_n + i\beta_n)\}]z^n$$
$$-(\alpha_n + i\beta_n)z^{n+1}$$

$$= \alpha_0 t + [(\alpha_1 t - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z + ... + (\alpha_l t - \alpha_{l-1})z^l + (\alpha_{l+1}t - \alpha_l)z^{l+1}$$

$$+ ...... + [(k\alpha_n t - \alpha_{n-1}) - (k\alpha_n t - \alpha_n t)]z^n - \alpha_n z^{n+1} + i[\beta_0 t + (\beta_1 t - \beta_0)z$$
$$+ ...... + (\beta_n t - \beta_{n-1})z^n - \beta_n z^{n+1}].$$

For $|z| = t$, we have by using the hypothesis

$$\begin{aligned}|F(z)| &\le |\alpha_0|t+|\tau\alpha_0-\alpha_0|t+|\alpha_1 t-\tau\alpha_0|t+|\alpha_2 t-\alpha_1|t^2+......+|\alpha_l t-\alpha_{l-1}|t^l+|\alpha_{l+1}t-\alpha_l|t^{l+1}\\ &\quad+......+|k\alpha_n t-\alpha_{n-1}|t^n+|k\alpha_n t-\alpha_n t|t^n+|\alpha_n|t^{n+1}+|\beta_0|t+(|\beta_1|t+|\beta_0|)t+......\\ &\quad+(|\beta_n|t+|\beta_{n-1}|)t^n+|\beta_n|t^{n+1}\\ &\le |\alpha_0|t+(1-\tau)|\alpha_0|t+(\alpha_1 t-\alpha_0)t+(\alpha_2 t-\alpha_1)t^2+......+(\alpha_l t-\alpha_{l+1})t^l\\ &\quad+(\alpha_l-\alpha_{l+1}t)t^{l+1}+......+(\alpha_{n-1}-k\alpha_n t)t^n+(1-k)|\alpha_n|t^{n+1}+|\alpha_n|t^{n+1}+2\sum_{j=0}^{n}|\beta_j|t^{j+1}\\ &= (2|\alpha_0|-\alpha_0)t-\tau|\alpha_0|t+2\alpha_l t^{l+1}+2|\alpha_n|t^{n+1}-k(|\alpha_n|+\alpha_n)t^{n+1}+2\sum_{j=0}^{n}|\beta_j|t^{j+1}\\ &= M_2.\end{aligned}$$

The theorem now follows as in the proof of Theorem 1.

Proof of Theorem 3: As in the proof of Theorem 2,

$$\begin{aligned}F(z) &= \alpha_0 t+[(\alpha_1 t-\tau_1\alpha_0)+(\tau_1\alpha_0-\alpha_0)]z+......+(\alpha_l t-\alpha_{l-1})z^l+(\alpha_{l+1}t-\alpha_l)z^{l+1}\\ &\quad+......+[(k_1\alpha_n t-\alpha_{n-1})-(k_1\alpha_n t-\alpha_n t)]z^n-\alpha_n z^{n+1}+i[\beta_0 t+\{(\beta_1 t-\tau_2\beta_0)\\ &\quad+(\tau_2\beta_0-\beta_0)\}z+......+(\beta_m t-\beta_{m-1})z^m+(\beta_{m+1}t-\beta_m)z^{m+1}+......\\ &\quad+\{(k_2\beta_n t-\beta_{n-1})-(k_2\beta_n t-\beta_n t)\}z^n-\beta_n z^{n+1}].\end{aligned}$$

For $|z|=t$, we have by using the hypothesis,

$$\begin{aligned}|F(z)| &\le |\alpha_0|t+|\tau\alpha_0-\alpha_0|t+|\alpha_1 t-\tau\alpha_0|t+|\alpha_2 t-\alpha_1|t^2+......+|\alpha_l t-\alpha_{l-1}|t^l+|\alpha_{l+1}t-\alpha_l|t^{l+1}\\ &\quad+......+|k_1\alpha_n t-\alpha_{n-1}|t^n+|k_1\alpha_n t-\alpha_n t|t^n+|\alpha_n|t^{n+1}+|\beta_0|t+|\beta_1 t-\tau_2\beta_0|t\\ &\quad+|\tau_2\beta_0-\beta_0|t+......+|\beta_m t-\beta_{m-1}|t^m+|\beta_{m+1}t-\beta_m|t^{m+1}+......+|k_2\beta_n t-\beta_{n-1}|t^n\\ &\quad+|k_2\beta_n t-\beta_n t|t^n+|\beta_n|t^{n+1}\\ &= |\alpha_0|t+(1-\tau_1)|\alpha_0|t+(\alpha_1 t-\tau_1\alpha_0)t+(\alpha_2 t-\alpha_1)t^2+......+(\alpha_l t-\alpha_{l-1})t^l\\ &\quad+(\alpha_l-\alpha_{l+1}t)t^{l+1}+......+(\alpha_{n-1}-k_1\alpha_n t)t^n+(1-k_1)|\alpha_n|t^{n+1}+|\beta_0|t+(1-\tau_2)|\beta_0|t\\ &\quad+(\beta_1 t-\tau_2\beta_0)t+(\beta_2 t-\beta_1)t^2+......+(\beta_m t-\beta_{m-1})t^m+(\beta_m-\beta_{m+1}t)t^{m+1}+......\\ &\quad+(\beta_{n-1}-k_2\beta_n t)t^n+(1-k_2)|\beta_n|t^n+|\beta_n|t^{n+1}\end{aligned}$$

$$= 2|\alpha_0|t - \tau_1(|\alpha_0| + \alpha_0)t + 2\alpha_l t^{l+1} + |\alpha_n|t^{n+1} - k_1(|\alpha_n| + \alpha_n)t^{n+1}$$

$$+ 2|\beta_0|t - \tau_1(|\beta_0| + \beta_0)t + 2\beta_m t^{m+1} + |\beta_n|t^{n+1} - k_2(|\beta_n| + \beta_n)t^{n+1}.$$

$$= M_3.$$

The result now follows as in the proof of Theorem 1.

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