IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 1 Issue 4, June 2014.

www.ijiset.com

ISSN 2348 - 7968

A*-Algebras over Matrices

K.Suguna Rao[1], P.Koteswara Rao[2] Dr.V.Amarendra Babu[3]

[1]Dept.of Mathematics ,Acharya Nagarjuna University, Nagarjuna Nagar , Andhra Pradesh, India-522510
[2]Professor of Mathematics,Dept.of Commerce,Acharya Nagarjuna University,,Andhra Pradesh,India-522 510
[3]Assistant Professor,Dept.of Mathematics,Acharya Nagarjuna University,Nagarjuna Nagar,Andhra Pradesh,India-522 510.

Abstract: In this paper we introduce a new concept A*-algebras over the matrices. We give the definition of A*-matrices, example, Proves the theorem the set of m×n matrices forms an A*-algebra, Boolean algebras over the matrices forms an A*-algebra ,Every finite product of a finite A*-algebras is an A*-algebra of matrices over 3= {0,1,2}.Congruence relation on A*-matrix ,some other theorems related to congruence and finally we introduce the definition of A*-ideal over the matrices and some other theorems.

Key words: A*-algebra, A*-congruence, A*-ideal, Boolean algebra.

1. Introduction

E.G. Manes introduced the concept of Ada (Algebra of disjoint alternatives) (A, $\wedge$,V, $(-)^{\sim}$,$(-)_{\pi}$, 0, 1,2) ,where $\wedge$,V are binary operations on A, $(-)^{\sim}$, $(-)_{\pi}$ are unary operations and 0,1,2 are distinguished elements on A. In 1993 E.G.Manes introduced a new definition of Ada of his later paper [9] Adas and the equational theory of if-then-else.Fernando Guzman and Craig C Squire [4] introduced the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C -algebras (A,$\wedge$,V,(-) ~)),where $\wedge$,V are binary operations on A,(-) ~ is a unary operation). In 1994, P.Koteswara Rao[6]first introduced the concept of A*-algebra (A, $\wedge$, V, *, (-) ~, $(-)_{\pi}$, 0, 1, 2))(where $\wedge$,V,* are binary operations on A, (-) ~,$(-)_{\pi}$ are unary operations and 0,1,2 are distinguished elements on A studied the equivalence with Ada, Calgebra,Ada's connection with 3-Ring, Stone type representation , the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra.

2. Main Part

2.1 Note: Every element of 3^B is a sequence of elements of 0,1,2. If B is a finite order n .Then every element of $3^B(=3^n)$ is an ordered n-tuple of 0,1,2 or n-vector.

A vector (or n-tuple) may be written in a column which is called column vector or in a row which is called a row vector .Every element in 3^n will be written in a row will be called row vector ,and every element in A^m, where A is an A*-algebra is written in a column(i.e) every element of A^m is a column vector or m-vector.

2.2 Note : The A*-algebras A^m and 3^n are called A*-algebras of n-tuples.

2.3 Definition: Suppose A is an A*-algebra. A matrix M who elements are elements of A is called a matrix over A.

2.4 Example: M = $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ is a matrx over 3= {0,1,2}.

2.5 Definition: Suppose I = {1,2,........,m} , J = {1,2,......,n}. Every m×n matrix over A is a function f: I×J→ A ,where f(i,j) = a_{ij} where $a_{ij} \in$ A, i= 1,2,.....,m , j = 1,2,.........,n.

$M = [a_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$.

2.6 Theorem: Suppose A is an A* -algebra. $\mathcal{M}$ is the set of all m × n matrices over A. Then ($\mathcal{M}$,∧ ,∗ ,(–)˜ ,$(–)_\pi$, 1) is an A*-algebra where ∧ ,∗, ,(–)˜ ,$(–)_\pi$, 1 are defined by

$A \wedge B = [a_{ij}] \wedge [b_{ij}] = [a_{ij} \wedge b_{ij}]$

$A * B = [a_{ij}] * [b_{ij}] = [a_{ij} * b_{ij}]$

$A^{\sim} = [a_{ij}]^{\sim} = [a_{ij}^{\sim}]$

$A_\pi = [a_{ij}]_\pi = [a_{ij\pi}]$

$1 = [a_{ij}]$ where $a_{ij} = 1$ for all i,j

$0 = [b_{ij}]$ where $b_{ij} = 0$ for all i , j

$2 = [c_{ij}]$ where $c_{ij} = 2$ for all i, j.

Proof:

(i) . $A = [a_{ij}]$ $A_\pi = [a_{ij\pi}]$

$A_\pi \vee (A_\pi)^{\sim} = [a_{ij\pi}] \vee [a_{ij\pi}^{\sim}] = [1_{ij}]$ where $1_{ij} = 1$ for every i,j.

Therefore $A_\pi \vee (A_\pi)^{\sim} = 1$.

$(A_\pi)_\pi = [a_{ij}]_{\pi\pi} = [a_{ij\pi}]_\pi = [(a_{ij\pi})_\pi] = [a_{ij\pi}] = A_\pi$

Therefore $(A_\pi)_\pi = A_\pi$.

(ii) . $A_\pi \vee B_\pi = [a_{ij}]_\pi \vee [b_{ij}]_\pi$

$= [a_{ij\pi}] \vee [b_{ij\pi}]$

$= [a_{ij\pi} \vee b_{ij\pi}]$

$= [b_{ij\pi} \vee a_{ij\pi}]$

$= [b_{ij}]_\pi \vee [a_{ij}]_\pi$

$= B_\pi \vee A_\pi$

Therefore $A_\pi \vee B_\pi = B_\pi \vee A_\pi$

(iii) $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$

$(A_\pi \vee B_\pi) \vee C_\pi = ([a_{ij}]_\pi \vee [b_{ij}]_\pi) \vee [c_{ij}]_\pi$

$= ([a_{ij}]_\pi \vee [b_{ij}]_\pi) \vee [c_{ij\pi}]$

$= [a_{ij\pi} \vee b_{ij\pi}] \vee [c_{ij\pi}]$

$= [(a_{ij\pi} \vee b_{ij\pi}) \vee c_{ij\pi}]$

$= [a_{ij\pi} \vee (b_{ij\pi} \vee c_{ij\pi})]$

$= [a_{ij\pi}] \vee [b_{ij\pi} \vee c_{ij\pi}]$

Therefore $(A_\pi \vee B_\pi) \vee C_\pi = A_\pi \vee (B_\pi \vee C_\pi)$

(iv) $(A_\pi \wedge B_\pi) \vee (A_\pi \wedge (B_\pi)^\sim) = ([a_{ij}]_\pi \wedge [b_{ij}]_\pi) \vee ([a_{ij}]_\pi \wedge [b_{ij}]^\sim_\pi)$

$= ([a_{ij\pi}] \wedge [b_{ij\pi}]) \vee ([a_{ij\pi}] \wedge [b_{ij\pi}^\sim])$

$= [(a_{ij\pi} \wedge b_{ij\pi}) \vee (a_{ij\pi} \wedge b_{ij\pi}^\sim)]$

$= [a_{ij\pi}] = [a_{ij}]_\pi = A_\pi$

Therefore $(A_\pi \wedge B_\pi) \vee (A_\pi \wedge (B_\pi)^\sim) = A_\pi$.

(v) $(A \wedge B)_\pi = ([a_{ij}] \wedge [b_{ij}])_\pi = [a_{ij} \wedge b_{ij}]_\pi$

$= [(a_{ij} \wedge b_{ij})_\pi] = [a_{ij\pi} \wedge b_{ij\pi}]$

$= [a_{ij\pi}] \wedge [b_{ij\pi}] = [a_{ij}]_\pi \wedge [b_{ij}]_\pi = A_\pi \wedge B_\pi$

Therefore $(A \wedge B)_\pi = A_\pi \wedge B_\pi$.

$(A \wedge B)^\# = ([a_{ij}] \wedge [b_{ij}])^\# = [a_{ij} \wedge b_{ij}]^\#$

$= [(a_{ij} \wedge b_{ij})^\#] = [a^\#_{ij} \vee b^\#_{ij}]$

$= [a_{ij}]^\# \vee [b_{ij}]^\# = A^\# \vee B^\#$.

Therefore $(A \wedge B)^\# = A^\# \vee B^\#$.

(vi) $A^\sim_\pi = [a_{ij}]^\sim_\pi = [a_{ij}^\sim]_\pi = [a_{ij}{}^\sim{}_\pi] = [(a_{ij\pi} \vee a_{ij}^\#)^\sim]$

$= [a_{ij\pi} \vee a_{ij}^\#]^\sim = ([a_{ij\pi}] \vee [a_{ij}^\#])^\sim$

$= ([a_{ij}]_\pi \vee [a_{ij}]^\#)^\sim = (A_\pi \vee A^\#)^\sim$.

Therefore $A^{\sim}{}_{\pi} = (A_{\pi} \vee A^{\#})^{\sim}$.

$A^{\sim\#} = [a_{ij}]^{\sim\#} = [a_{ij}{}^{\sim}]^{\#} = [a_{ij}{}^{\sim\#}] = [a_{ij}{}^{\#}] = [a_{ij}]^{\#} = A^{\#}$.

Therefore $A^{\sim\#} = A^{\#}$.

(vii) $(A * B)_{\pi} = ([a_{ij}] * [b_{ij}])_{\pi} = [(a_{ij} * b_{ij})_{\pi}]$

$= [a_{ij\pi}] = [a_{ij}]_{\pi} = A_{\pi}$

Therefore $(A * B)_{\pi} = A_{\pi}$.

(viii) $A = B \Leftrightarrow [a_{ij}] = [b_{ij}] \Leftrightarrow a_{ij} = b_{ij}$ for every i, j.

$\Leftrightarrow a_{ij\pi} = b_{ij\pi}$, $a_{ij}{}^{\#} = b_{ij}{}^{\#}$ for every i, j.

$\Leftrightarrow [a_{ij}]_{\pi} = [b_{ij}]_{\pi}$, $[a_{ij}]^{\#} = [b_{ij}]^{\#}$

$\Leftrightarrow A_{\pi} = B_{\pi}$, $A^{\#} = B^{\#}$.

Therefore $A = B \Leftrightarrow A_{\pi} = B_{\pi}$, $A^{\#} = B^{\#}$.

Therefore by the above all conditions $(\mathcal{M}, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra .

2.7 Theorem: Let $(B, \wedge, (-)^{\sim}, 0)$ be a Boolean algebra over the matrices, then $\mathcal{A}(B) = \{ (A, B) / A = [a_{ij}], B = [b_{ij}] \in B, A \wedge B = 0\}$ is a A*- algebra over the matrices. Here A*-algebra operations $\wedge, \vee, *, (-)^{\sim}, (-)_{\pi}$ are defined as follows.

For $A = [a_{ij\pi}, a_{ij}{}^{\#}]$, $B = [b_{ij\pi}, b_{ij}{}^{\#}] \in \mathcal{A}(B)$

(i) $A \wedge B = [a_{ij\pi}\, b_{ij\pi}, a_{ij\pi} b_{ij}{}^{\#} + a_{ij}{}^{\#} b_{ij\pi} + a_{ij}{}^{\#}\, b_{ij}{}^{\#}]$, where juxtaposition , +, $(-)^{\sim}$ are respectively $\wedge, \vee, (-)^{\sim}$ in Boolean algebra B.

(ii) $A \vee B = [a_{ij\pi}\, b_{ij\pi} + a_{ij\pi} b_{ij}{}^{\#} + a_{ij}{}^{\#} b_{ij\pi}, a_{ij}{}^{\#}\, b_{ij}{}^{\#}]$

(iii) $A^{\sim} = [a_{ij}{}^{\#}, a_{ij\pi}]$

(iv) $A_{\pi} = [a_{ij\pi}, (a_{ij\pi})^{\sim}]$

(v) $A * B = [a_{ij\pi}, (a_{ij\pi})^{\sim} b_{ij}{}^{\#}]$

(vi) $1 = [1_{ij}, 0_{ij}]$, $0 = [0_{ij}, 1_{ij}]$.

Proof: clearly $(A^{\sim} \wedge B^{\sim})^{\sim} = (A^{\sim})^{\sim} \vee (B^{\sim})^{\sim} = A \vee B$.

$\mathcal{B}(\mathcal{A}(B)) = \{ A_{\pi} / A \in \mathcal{A}(B) \} = \{ [a_{ij\pi}, (a_{ij\pi})^{\sim}] / a_{ij\pi} \in B \}$

Therefore $\mathcal{B}(\mathcal{A}(B)) = B$.

Therefore (i) through (iv) in this theoem hold in $\mathcal{A}$(B).

$(A \wedge B)_\pi = [a_{ij\pi}\ b_{ij\pi},\ a_{ij\pi} b_{ij}^{\#} + a_{ij}^{\#} b_{ij\pi} + a_{ij}^{\#}\ b_{ij}^{\#}]_\pi$

$= [a_{ij\pi}\ b_{ij\pi},\ a_{ij\pi}^{\sim} + b_{ij\pi}^{\sim}]$

$A_\pi \wedge B_\pi = [a_{ij\pi},\ a_{ij\pi}^{\sim}] \wedge [b_{ij\pi},\ b_{ij\pi}^{\sim}]$

$= [a_{ij\pi}\ b_{ij\pi},\ a_{ij\pi} b_{ij\pi}^{\sim} + a_{ij\pi}^{\sim} b_{ij\pi} + a_{ij\pi}^{\sim}\ b_{ij\pi}^{\sim}]$

$= [a_{ij\pi}\ b_{ij\pi},\ a_{ij\pi}^{\sim} + b_{ij\pi}^{\sim}]$

Therefore $(A \wedge B)_\pi = A_\pi \wedge B_\pi$.

$(A \wedge B)_\pi^{\sim} = (A^{\sim} \vee B^{\sim})_\pi$

$= [a_{ij\pi}\ b_{ij}^{\#} + a_{ij}^{\#} b_{ij\pi} + a_{ij}^{\#} b_{ij}^{\#},\ a_{ij\pi} b_{ij\pi}]_\pi$

$= [a_{ij\pi}\ b_{ij}^{\#} + a_{ij}^{\#} b_{ij\pi} + a_{ij}^{\#} b_{ij}^{\#},\ (a_{ij\pi}^{\sim} + b_{ij}^{\#\sim})\,(a_{ij}^{\#\sim} + b_{ij\pi}^{\sim})\,(a_{ij}^{\#\sim} + b_{ij}^{\#\sim})]$.

$(A^{\sim} \wedge B)_\pi \vee (A \wedge B^{\sim})_\pi \vee (A^{\sim} \wedge B^{\sim})_\pi$

$= [a_{ij}^{\#} b_{ij\pi},\ a_{ij}^{\#\sim} + b_{ij\pi}^{\sim}] \vee [a_{ij\pi}\ b_{ij}^{\#},\ a_{ij\pi}^{\#\sim} + b_{ij}^{\sim}] \vee [a_{ij}^{\#} b_{ij}^{\#},\ a_{ij}^{\#\sim} + b_{ij}^{\#\sim}]$

$= [a_{ij\pi}\ b_{ij}^{\#} + a_{ij}^{\#} b_{ij\pi} + a_{ij}^{\#} b_{ij}^{\#},\ (a_{ij\pi}^{\sim} + b_{ij}^{\#\sim})\,(a_{ij}^{\#\sim} + b_{ij\pi}^{\sim})\,(a_{ij}^{\#\sim} + b_{ij}^{\#\sim})]$.

Therefore $(A \wedge B)_\pi^{\sim} = (A^{\sim} \wedge B)_\pi \vee (A \wedge B^{\sim})_\pi \vee (A^{\sim} \wedge B^{\sim})_\pi$

Since $A^{\#} = (A_\pi \vee (A^{\sim})_\pi)^{\sim}$.

Therefore $A^{\#} = [a_{ij\pi}^{\sim}\ a_{ij}^{\#\sim},\ a_{ij\pi} + a_{ij}^{\#}]$.

Clearly $A^{\sim\#} = A^{\#}$.

$(A \wedge B)^{\#\sim} = (A \wedge B)_\pi \vee (A \wedge B)^{\sim}{}_\pi$

$= (A \wedge B)_\pi \vee (A^{\sim} \wedge B)_\pi \vee (A \wedge B^{\sim})_\pi \vee (A^{\sim} \wedge B^{\sim})_\pi$

$= A_\pi \wedge (B_\pi \vee B^{\sim}{}_\pi) \vee A^{\sim} (B^{\sim}{}_\pi B^{\sim}{}_\pi)$

$= (A_\pi \vee A^{\sim}{}_\pi) \wedge (B^{\sim}{}_\pi \vee B^{\sim}{}_\pi)$

$= A^{\#\sim} \wedge B^{\#\sim}$.

Therefore $(A \wedge B)^{\#} = A^{\#} \vee B^{\#}$.

Clearly $A_\pi^{\sim} = [a_{ij}^{\#},\ a_{ij}^{\#\sim}]$.

$(A_\pi \vee A^{\#})^{\sim} = [a_{ij\pi},\ a_{ij\pi}^{\sim}] \vee [a_{ij}{}^{\sim}{}_\pi\, a_{ij}^{\#\sim},\ (a_{ij\pi} + b_{ij}^{\#})^{\sim}]$.

Therefore $A_\pi^{\sim} = (A_\pi \vee A^{\#})^{\sim}$.

IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 1 Issue 4, June 2014.
www.ijiset.com
ISSN 2348 - 7968

Since $A * B = [a_{ij\pi}, (a_{ij\pi})^{\sim} b_{ij}^{\#}]$.

Therefore $(A * B)_{\pi} = A_{\pi}$.

$(A * B)^{\#} = [a_{ij\pi}, (a_{ij\pi})^{\sim} b_{ij}^{\#}]^{\#}$.

$= [a_{ij\pi}^{\sim} (a_{ij\pi} + b_{ij}^{\#\sim}), a_{ij\pi} + (a_{ij\pi})^{\sim} b_{ij}^{\#}]$.

$= [(a_{ij\pi})^{\sim} b_{ij}^{\#\sim}, a_{ij\pi} + b_{ij}^{\#}]$.

Therefore $(A * B)^{\#} = [(a_{ij\pi})^{\sim} b_{ij}^{\#\sim}, a_{ij\pi} + b_{ij}^{\#}]$.

$(A_{\pi})^{\sim} \wedge (B_{\pi}^{\sim})^{\sim} = [a_{ij\pi}^{\sim}, a_{ij\pi}] \wedge [b_{ij}^{\#\sim}, b_{ij}^{\#}]$

$= [a_{ij\pi}^{\sim} b_{ij}^{\#\sim}, a_{ij\pi} + b_{ij}^{\#}]$.

Therefore $(A_{\pi})^{\sim} \wedge (B_{\pi}^{\sim})^{\sim} = [a_{ij\pi}^{\sim} b_{ij}^{\#\sim}, a_{ij\pi} + b_{ij}^{\#}]$.

Therefore $(A * B)^{\#} = (A_{\pi})^{\sim} \wedge (B_{\pi}^{\sim})^{\sim}$.

$A = B \Rightarrow A_{\pi} = B_{\pi},\ A^{\#} = B^{\#}$ is clear .

Suppose $A_{\pi} = B_{\pi},\ A^{\#} = B^{\#}$

Since $A_{\pi} = B_{\pi} = [a_{ij\pi}, (a_{ij\pi})^{\sim}] = [b_{ij\pi}, (b_{ij\pi})^{\sim}] \Rightarrow [a_{ij\pi}] = [b_{ij\pi}]$

Since $A_{\pi} = B_{\pi},\ A^{\#} = B^{\#} \Rightarrow A_{\pi}^{\sim} = B_{\pi}^{\sim}$.

$\Rightarrow [a_{ij}^{\#}, a_{ij}^{\#\sim}] = [b_{ij}^{\#}, b_{ij}^{\#\sim}]$

$\Rightarrow a_{ij}^{\#} = b_{ij}^{\#}$.

Therefore $[a_{ij\pi}, a_{ij}^{\#}] = [b_{ij\pi}, b_{ij}^{\#}]$

$\Rightarrow A = B$.

Therefore $\mathcal{A}(B)$ is an A*-algebra over the matrices.

2.8 Note: Suppose A is a finite A*-algebra then thereexist a finite set X such that A is isomorphic to a sub A*-algebra of 3^{x} .Therefore A is isomorphic to an A*-algebra of n-tuples of 0,1,2.Every element of A is a n-row vector of elements 0,1,2.A^{m} is an A*-algebra .Every element of A^{m} is m-column vector of elements of A.

2.9 Theorem : Every finite product of a finite A*-algebras is an A*-algebra of matrices over 3= {0,1,2}.

Proof: Suppose A is a finite A*-algebra .there exists a finite X(=B(X)) such that A is isomorphic to a subalgebra of 3^{x}. Suppose mod X = n.

Therefore every element of A is n-row vector of elements 0,1,2.

Suppose m is a positive integer.

$\Rightarrow A^m$ is an A*-algebra.

Every element a of A^m is m-column vector, $a^T = [a_1 , a_2,.........,a_m]$ of elements $a_1, a_2, ..., a_m$ (Since a_i is n-row vector of elements 0,1,2.

$a = [a_{ij}]$, where $a_{ij} = 0$ or 1 or 2 , is a m×n matrix over 3= {0,1,2}.

Suppose $a,b,c \in A^m$.

$\Rightarrow a = [a_{ij}]$, $b = [b_{ij}]$, $c = [c_{ij}]$ are m×n matrix over 3.Then $a \wedge b = [a_{ij} \wedge b_{ij}]$ $a * b = [a_{ij}*b_{ij}]$ $a^{\sim} = [a_{ij}^{\sim}]$, $a_\pi = [a_{ij\pi}]$,$1 = [1_{ij}]$,$0 = [0_{ij}]$, $2= [2_{ij}]$, where $1_{ij} = 1$, $0_{ij} = 0$, $2_{ij} = 2$ for all matices over 3.

Therefore A is isomorphic to a sub A*-algebra of A*-algebra over 3= {0,1,2}.

2.10 Definition: A relation $\overset{c}{-}$ on an A*-matrix is an equivalence relation on $\mathcal{M}$ satisfies the following.

(i)Reflexive: $(A,A) \in \overset{c}{-}$, for all $A\in \mathcal{M}$

(ii) Symmetric: $(A, B) \in \overset{c}{-} \Rightarrow (B,A) \in \overset{c}{-}$,for all $A,B\in \mathcal{M}$

(iii)Transitive: $(A,B) \in \overset{c}{-}$ and $(B,C) \in \overset{c}{-} \Rightarrow (A,C) \in \overset{c}{-}$,for all $A,B,C \in \mathcal{M}$

(iv) $A \overset{c}{-} B \Rightarrow A_\pi \overset{c}{-} B_\pi$, $A^{\#} \overset{c}{-} B^{\#}$, $A^{\sim} \overset{c}{-} B^{\sim}$, for all $A,B \in \mathcal{M}$

(v) $A \overset{c}{-} B$, $C \overset{c}{-} D \Rightarrow (A* C) \overset{c}{-} (B * D)$, $(A \wedge C) \overset{c}{-} (B \wedge D)$, for all $A,B,C,D \in \mathcal{M}$

2.11 Note : We write $A \overset{c}{-} B$ to indicate $(A,B) \in \overset{c}{-}$.

2.12 Definition: Let $\mathcal{M}$ be a A*-matix. Then the set of all congruences on $\mathcal{M}$ is denoted by Con($\mathcal{M}$).

2.13Theorem: Let $(\mathcal{M}, \wedge,* ,(-)^{\sim} , (-)_\pi ,0)$ be an A*-algebra over the matriceses and $\overset{c}{-}$ be a congruence relation on B= B($\mathcal{M}$). The relation $\overset{c}{-}'$ is a congruence relation on $\mathcal{M}$. Where $\overset{c}{-}'$ on $\mathcal{M}$ is defined by $A \overset{c}{-}' B \Leftrightarrow A_\pi \overset{c}{-} B_\pi$, $A^{\#} \overset{c}{-} B^{\#}$.

Proof: Let $(\mathcal{M}, \wedge,* ,(-)^{\sim} , (-)_\pi ,0)$ be an A*-algebra over the matriceses and $\overset{c}{-}$ be a congruence relation on B= B($\mathcal{M}$).

Claim: To show that $\overset{c}{-}'$ is a congruence relation on $\mathcal{M}$.

Let $A \in \mathcal{M} \Rightarrow A_\pi \overset{c}{-} A_\pi$, $A^{\#} \overset{c}{-} A^{\#}$

$\Rightarrow A \overset{c}{-}' A$.

This implies $A \overset{c}{=}' A$, for $A \in \mathcal{M}$.

Therefore $\overset{c}{=}'$ is reflexive.

Suppose $A \overset{c}{=}' B \Rightarrow A_\pi \overset{c}{=}' B_\pi$, $A^\# \overset{c}{=}' B^\#$

$\Rightarrow B_\pi \overset{c}{=}' A_\pi$, $B^\# \overset{c}{=} A^\#$

$\Rightarrow B \overset{c}{=}' A$.

This implies $A \overset{c}{=}' B \Rightarrow B \overset{c}{=}' A$.

Therefore $\overset{c}{=}'$ is a symmetric.

Suppose $A \overset{c}{=}' B$ and $B \overset{c}{=}' C \Rightarrow A_\pi \overset{c}{=} B_\pi$ and $B_\pi \overset{c}{=} C_\pi$, $A^\# \overset{c}{=} B^\#$ and $B^\# \overset{c}{=} C^\#$

$\Rightarrow A_\pi \overset{c}{=} C_\pi$, $A^\# \overset{c}{=} C^\#$

$\Rightarrow A \overset{c}{=}' C$.

This implies $A \overset{c}{=}' B$, $B \overset{c}{=}' C \Rightarrow A \overset{c}{=}' C$

Therefore $\overset{c}{=}'$ is transitive.

Therefore $\overset{c}{=}'$ is an equivalence relation on $\mathcal{M}$.

Suppose $A \overset{c}{=}' B$ and $C \overset{c}{=}' D \Rightarrow A_\pi \overset{c}{=} B_\pi$ and $A^\# \overset{c}{=} B^\#$, $C_\pi \overset{c}{=} D_\pi$,$C^\# \overset{c}{=} D^\#$

$\Rightarrow(A_\pi \wedge C_\pi) \overset{c}{=} (B_\pi \wedge D_\pi)$, $(A^\# \vee C^\#) \overset{c}{=} (B^\# \vee D^\#)$ (Since $\overset{c}{=}$ is a congruence relation on $\mathcal{M}$)

$\Rightarrow (A \wedge C)_\pi \overset{c}{=} (B \wedge D)_\pi$, $(A \wedge C)^\# \overset{c}{=} (B \wedge D)^\#$

$\Rightarrow (A \wedge C) \overset{c}{=}' (B \wedge D)$.

Therefore $A \overset{c}{=}' B$ and $C \overset{c}{=}' D \Rightarrow (A \wedge C) \overset{c}{=}' (B \wedge D)$.

Suppose $A \overset{c}{=}' B \Rightarrow A_\pi \overset{c}{=} B_\pi$, $A^\# \overset{c}{=} B^\#$

Therefore $(A_\pi)_\pi \overset{c}{=} (B_\pi)_\pi$

Since $0 \overset{c}{=} 0 \Rightarrow (A_\pi)^\# \overset{c}{=} (B_\pi)^\#$.

Therefore $(A_\pi)_\pi \overset{c}{=} (B_\pi)_\pi$, $(A_\pi)^\# \overset{c}{=} (B_\pi)^\# \Rightarrow A_\pi \overset{c}{=}' B_\pi$ (Since by the definition of $\overset{c}{=}'$)

Therefore $A \overset{c}{=}' B \Rightarrow A_\pi \overset{c}{=}' B_\pi$.

Suppose $A \overset{c}{=}' B$, $C \overset{c}{=}' D \Rightarrow A_\pi \overset{c}{=} B_\pi$, $A^\# \overset{c}{=} B^\#$,$C_\pi \overset{c}{=} D_\pi$,$C^\# \overset{c}{=} D^\#$.

ISSN 2348 - 7968

Since $A_\pi \overset{c}{=} B_\pi \Rightarrow (A * C)_\pi \overset{c}{=} (B * D)_\pi$.

Since $A^{\#} \overset{c}{=} B^{\#}$, $0 \overset{c}{=} 0 \Rightarrow A^{\#}{}_\pi \overset{c}{=} B^{\#}{}_\pi$, $A^{\#\#} \overset{c}{=} B^{\#\#}$.

$\Rightarrow A^{\#} \overset{c}{=}' B^{\#}$.

Therefore $A^{\#} \overset{c}{=} B^{\#} \Rightarrow A^{\#} \overset{c}{=}' B^{\#}$.

Now $(A_\pi \vee A^{\#}) \overset{c}{=} (B_\pi \vee B^{\#}) \Rightarrow (A^{\sim}{}_\pi)^{\sim} \overset{c}{=} (B^{\sim}{}_\pi)^{\sim}$

$\Rightarrow A^{\sim}{}_\pi \overset{c}{=} B^{\sim}{}_\pi$

$A^{\#} \overset{c}{=} B^{\#} \Rightarrow A^{\sim\#} \overset{c}{=} B^{\sim\#}$ (Since $A^{\sim\#} = A^{\#}$).

$A \overset{c}{=}' B \Rightarrow (A^{\sim})_\pi \overset{c}{=} (B^{\sim})_\pi$, $(A^{\sim})^{\#} \overset{c}{=} (B^{\sim})^{\#} \Rightarrow A^{\sim} \overset{c}{=} B^{\sim}$.

Finally to prove that if $A \overset{c}{=}' B$, $C \overset{c}{=}' D \Rightarrow (A * C) \overset{c}{=}' (B * D)$.

Suppose $A \overset{c}{=}' B$, $C \overset{c}{=}' D \Rightarrow A_\pi \overset{c}{=} B_\pi$, $C_\pi \overset{c}{=} D_\pi$, $A^{\#} \overset{c}{=} B^{\#}$, $C^{\#} \overset{c}{=} D^{\#}$.

Clearly $A_\pi \overset{c}{=} B_\pi \Rightarrow (A * C)_\pi \overset{c}{=} (B * D)_\pi$.

Since $(A_\pi)^{\sim} \wedge (C_\pi{}^{\sim})^{\sim} \overset{c}{=} (B_\pi)^{\sim} \wedge (D_\pi{}^{\sim})^{\sim} \Rightarrow (A * C)^{\#} \overset{c}{=} (B * D)^{\#}$.

Therefore $(A * C) \overset{c}{=}' (B * D)$.

Therefore $\overset{c}{=}'$ is a congruence relation on A.

2.14 Theorem: Let $\mathcal{M}$ be an A*-algebra over the matriceses and $\overset{c}{=}$ be a congruence relation on $\mathcal{M} \Leftrightarrow \overset{c}{=}_B$ is a congruence on B = B($\mathcal{M}$) ,Boolean algebra and $\overset{c}{=}'_B = \overset{c}{=}$.

Proof: Suppose $\overset{c}{=}$ is a congruence relation on $\mathcal{M}$

$\Rightarrow \overset{c}{=}_B$ is a Boolean congruence on B.

Clearly $\overset{c}{=} \subseteq \overset{c}{=}'_B$.

$A \overset{c}{=}'_B B \Leftrightarrow A_\pi \overset{c}{=}_B B_\pi$, $A^{\#} \overset{c}{=}_B B^{\#}$

$\Leftrightarrow A_\pi \overset{c}{=} B_\pi$, $A^{\#} \overset{c}{=} B^{\#}$

$\Leftrightarrow A \overset{c}{=} B$.

Therefore $\overset{c}{=}'_B = \overset{c}{=}$.

2.15 Theorem: If $(\mathcal{M}, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ is an A*-algebra and $\overset{c}{=}$ be a congruence relation on $\mathcal{M}$,then $A \overset{c}{=} B \Leftrightarrow$(i) $A_\pi(B_\pi)^{\sim} \overset{c}{=} 0$

(ii) $A_{\pi}{}^{\sim}(B_{\pi}{}^{\sim}) \stackrel{c}{=} 0.$

(iii) $(A_{\pi})^{\sim} B_{\pi} \stackrel{c}{=} 0.$

(iv) $(A_{\pi}{}^{\sim})^{\sim} B_{\pi}{}^{\sim} \stackrel{c}{=} 0.$

Proof: $(\mathcal{M}, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra and $\stackrel{c}{=}$ be a congruence relation on $\mathcal{M}$.

Suppose $A \stackrel{c}{=} B \Rightarrow A_{\pi} \stackrel{c}{=} B_{\pi}$, $A^{\#} \stackrel{c}{=} B^{\#}$, $A^{\sim} \stackrel{c}{=} B^{\sim}$, $A_{\pi} \stackrel{c}{=} B_{\pi}$, $(A_{\pi})^{\sim} \stackrel{c}{=} (B_{\pi})^{\sim}$

$\Rightarrow A_{\pi} (B_{\pi})^{\sim} \stackrel{c}{=} 0$

This implies (i).

Interchanging A, B in (i) and (ii) we have (iii) $(A_{\pi})^{\sim} B_{\pi} \stackrel{c}{=} 0$.and (iv) $(A_{\pi}{}^{\sim})^{\sim} B_{\pi}{}^{\sim} \stackrel{c}{=} 0$

Conversely assume (i) , (ii) , (iii) and (iv) .

Claim: To show that $A \stackrel{c}{=} B$:

First we have to prove that for any $A,B \in B$, $A B^{\sim} \stackrel{c}{=} 0 \Leftrightarrow A B \stackrel{c}{=} A$.

Suppose $A B^{\sim} \stackrel{c}{=} 0$.

$A = A \wedge I = A \wedge (B \vee B^{\sim}) = AB \vee AB^{\sim}$.

Therefore $A = AB \vee AB^{\sim}$.

$A B^{\sim} \stackrel{c}{=} 0 \Rightarrow AB \vee AB^{\sim} \stackrel{c}{=} A B \Rightarrow A \stackrel{c}{=} AB$.

Therefore $A B^{\sim} \stackrel{c}{=} 0 \Rightarrow A \stackrel{c}{=} AB$.

Conversely suppose that $A \stackrel{c}{=} AB$,i.e., $AB \stackrel{c}{=} A$.

$ABB^{\sim} \stackrel{c}{=} AB^{\sim} \Rightarrow 0 \stackrel{c}{=} AB^{\sim} \Rightarrow AB^{\sim} \stackrel{c}{=} 0$.

Therefore $A \stackrel{c}{=} AB \Rightarrow AB^{\sim} \stackrel{c}{=} 0$.

Hence $AB^{\sim} \stackrel{c}{=} 0 \Leftrightarrow AB \stackrel{c}{=} A$.

From $A_{\pi} B_{\pi}{}^{\sim} \stackrel{c}{=} 0 \Rightarrow A_{\pi} B_{\pi} \stackrel{c}{=} A_{\pi}$ (Since by above result).

By interchanging A,B we have $\Rightarrow A_{\pi} B_{\pi} \stackrel{c}{=} B_{\pi}$.

Therefore $A_{\pi} \stackrel{c}{=} B_{\pi}$ (By transitivity).

$(A_{\pi})^{\sim} B_{\pi} \stackrel{c}{=} 0 \Rightarrow (A_{\pi})^{\sim} (A_{\pi}{}^{\sim})^{\sim} B_{\pi} \stackrel{c}{=} 0$

$\Rightarrow A^{\#} B_{\pi} \stackrel{c}{=} 0$.

ISSN 2348 - 7968

$(A_\pi^{\sim})^{\sim} B_\pi \overset{c}{=} 0 \Rightarrow (A_\pi)^{\sim} (A_\pi^{\sim})^{\sim} B_\pi \overset{c}{=} 0 \Rightarrow A^{\#} B_\pi^{\sim} \overset{c}{=} 0$.

Taking $\vee$ for $A^{\#} B_\pi \overset{c}{=} 0$ and $A^{\#} B_\pi^{\sim} \overset{c}{=} 0 \Rightarrow A^{\#} (B_\pi \vee B_\pi^{\sim}) \overset{c}{=} 0$

$\Rightarrow A^{\#} B^{\#\sim} \overset{c}{=} 0 \Rightarrow A^{\#} B^{\#} \overset{c}{=} A^{\#}$ (Since by above result).

Similarly $A^{\#} B^{\#} \overset{c}{=} B^{\#} \Rightarrow A^{\#} \overset{c}{=} B^{\#}$ (Since by transitivity).

Therefore $(A_\pi * A^{\#}) \overset{c}{=} (B_\pi * B^{\#})$.

Hence $A \overset{c}{=} B$.

2.16 Definition: Let J be the non empty subset of an A*-algebra over the matrices. J is said to be M*-ideal if

(i) $B, C \in J \Rightarrow B \vee C, B * C \in J$.

(ii) $B \in J \Rightarrow B_\pi, B^{\#} \in J$.

(iii) $B \in J, C \in A \Rightarrow B_\pi C_\pi, B^{\#} C^{\#} \in J$.

2.17 Note: 5.14 is equivalent to the following:

(i) $B, C \in J \Rightarrow B \vee C, B * C \in J$
(ii) $B \in J, C \in A \Rightarrow B_\pi C_\pi, B^{\#} C^{\#} \in J$.

2.18 Theorem: Suppose $\overset{c}{=}$ is a congruence relation .Then $\overset{c}{=}(0)$ is M*-ideal.

Proof: Let $A \in \overset{c}{=}(0) \Rightarrow A \overset{c}{=} 0 \Rightarrow [a_{ij}] \overset{c}{=} [0_{ij}]$

$\Rightarrow [a_{ij}]_\pi \overset{c}{=} [0_{ij}]_\pi \Rightarrow [a_{ij}]_\pi \overset{c}{=} [0_{ij}] \Rightarrow A_\pi \overset{c}{=} 0 \Rightarrow A_\pi \in \overset{c}{=}(0)$.

Suppose $A \overset{c}{=} 0 \Rightarrow [a_{ij}] \overset{c}{=} [0_{ij}] \Rightarrow [a_{ij}]^{\#} \overset{c}{=} [0_{ij}]^{\#}$

$\Rightarrow [a_{ij}]^{\#} \overset{c}{=} [0_{ij}] \Rightarrow A^{\#} \overset{c}{=} 0 \Rightarrow A^{\#} \in 0$.

Let $A, B \in \overset{c}{=}(0) \Rightarrow A \overset{c}{=} 0, B \overset{c}{=} 0$.

$\Rightarrow [a_{ij}] \overset{c}{=} [0_{ij}], [b_{ij}] \overset{c}{=} [0_{ij}] \Rightarrow [a_{ij}] \vee [b_{ij}] \overset{c}{=} [0_{ij}], [a_{ij}] * [b_{ij}] \overset{c}{=} [0_{ij}]$

$\Rightarrow A \vee B \overset{c}{=} 0, A * B \overset{c}{=} 0$.

$\Rightarrow A \vee B \in \overset{c}{=}(0)$ and $A * B \in \overset{c}{=}(0)$.

Let $A \in \overset{c}{=}(0), B \in M \Rightarrow A \overset{c}{=} 0$ and $B \in M$

$\Rightarrow [a_{ij}] \overset{c}{=} [0_{ij}], [b_{ij}] \in M$.

$\Rightarrow [a_{ij}]_\pi [b_{ij}]_\pi \overset{c}{=} [0_{ij}] \Rightarrow [a_{ij} \wedge b_{ij}]_\pi \overset{c}{=} [0_{ij}] \Rightarrow (A \wedge B)_\pi \overset{c}{=} 0 \Rightarrow (A \wedge B)_\pi \in (0)$.

Clearly $A^{\#} B^{\#} \in \overset{c}{=} (0)$.

Therefore $\overset{c}{=} (0)$ is M*-ideal.

2.19 Theorem: Let $\mathcal{M}$ be an A*-algebras over the matriceses and J be $\mathcal{M}^{*}$-ideal of $\mathcal{M}$.Define $\overset{c}{=}$ on $\mathcal{M}$ by $A \overset{c}{=} B \Leftrightarrow A^{\#} (B^{\#})^{\sim}, A^{\#\sim} (B^{\#\sim})^{\sim}, A^{\#\sim} B^{\#}, (A^{\#\sim})^{\sim} B^{\#\sim}$. Then $\overset{c}{=}$ is a congruence relation on $\mathcal{M}$.

Proof: Let $(\mathcal{M}, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ be an A*-algebra over the matrices.

Suppose $A \in \mathcal{M}$

Since $A^{\#} A^{\#\sim} = 0$, $A^{\#\sim} (A^{\#\sim})^{\sim} = 0$.

We have that $A^{\#} (A^{\#})^{\sim}, A^{\#\sim} (A^{\#\sim})^{\sim}, A^{\#\sim} A^{\#}, (A^{\#\sim})^{\sim} A^{\#\sim} \in J$.

$\Rightarrow A \overset{c}{=} A$.

Therefore $\overset{c}{=}$ is reflexive.

Suppose $A \overset{c}{=} B \Rightarrow A^{\#} (B^{\#})^{\sim}, A^{\#\sim} (B^{\#\sim})^{\sim}, A^{\#\sim} B^{\#}, (A^{\#\sim})^{\sim} B^{\#\sim} \in J$.

$\Rightarrow B^{\#} (A^{\#})^{\sim}, B^{\#\sim} (A^{\#\sim})^{\sim}, B^{\#\sim} A^{\#}, (B^{\#\sim})^{\sim} A^{\#\sim} \in J$.

$\Rightarrow B \overset{c}{=} A$.

Therefore $\overset{c}{=}$ is symmetric.

Suppose $A \overset{c}{=} B$ and $B \overset{c}{=} C$.

$\Rightarrow A^{\#} (B^{\#})^{\sim}, A^{\#\sim} (B^{\#\sim})^{\sim}, A^{\#\sim} B^{\#}, (A^{\#\sim})^{\sim} B^{\#\sim}$ and $B^{\#} (C^{\#})^{\sim}, B^{\#\sim} (C^{\#\sim})^{\sim}, B^{\#\sim} C^{\#}, (B^{\#\sim})^{\sim} C^{\#\sim}$.

Now $A^{\#} (C^{\#})^{\sim} = A^{\#} (B^{\#} \vee B^{\#\sim}) C^{\#\sim}$

$= A^{\#} B^{\#} C^{\#\sim} \vee A^{\#} B^{\#\sim} C^{\#\sim}$

$= [A^{\#} B^{\#} C^{\#\sim}] \vee [C^{\#\sim} A^{\#} B^{\#\sim}] \in J$.

Therefore $A^{\#} (C^{\#})^{\sim} \in J$.

Similarly $A^{\#} (C^{\#})^{\sim}, A^{\#\sim} (C^{\#\sim})^{\sim}, A^{\#\sim} C^{\#}, (A^{\#\sim})^{\sim} C^{\#\sim} \in J$.

$\Rightarrow A \overset{c}{=} C$.

Therefore $\overset{c}{=}$ is transitive.

First we have to show that $A \overset{c}{=} B \Rightarrow A \wedge C \overset{c}{=} B \wedge C$, for all $C \in \mathcal{M}$

Suppose $A \overset{c}{=} B$

$\Rightarrow A^{\#}(B^{\#})^{\sim}, A^{\#\sim}(B^{\#\sim})^{\sim}, A^{\#\sim}B^{\#}, (A^{\#\sim})^{\sim}B^{\#\sim} \in J$

$(A \wedge C)^{\#}((B\wedge C)^{\#})^{\sim} = A^{\#}C^{\#}(B^{\#\sim} \vee C^{\#\sim}) \in J.$

Therefore $(A \wedge C)^{\#}\ ((B\wedge C)^{\#})^{\sim} \in J.$

Similarly $((A \wedge C)^{\#})^{\sim}\ (B\wedge C)^{\#} \in J.$

Now $((A \wedge C)^{\sim})^{\#}\ ((B\wedge C)^{\sim\#})^{\sim} = [(A^{\sim} \wedge C)^{\#} \vee (A \wedge C^{\sim})^{\#} \vee (A^{\sim} \wedge C^{\sim})^{\#}]$

$[(B^{\sim} \wedge C)^{\#} \vee (B \wedge C^{\sim})^{\#} \vee (B^{\sim} \wedge C^{\sim})^{\#}]^{\sim}.$

$= [A^{\#\sim}C^{\#} \vee A^{\#}C^{\#\sim} \vee A^{\#\sim}C^{\#\sim}]\ [B^{\#\sim}C^{\#} \vee B^{\#}C^{\#\sim} \vee B^{\#\sim}C^{\#\sim}]$

$=[A^{\#}C^{\#\sim} \vee A^{\#\sim}C^{\#} \vee A^{\#\sim}C^{\#\sim}]\ [(B^{\#})^{\sim} \vee (C^{\#\sim})^{\sim}] \vee [(B^{\#\sim})^{\sim} \vee (C^{\#})^{\sim}]\ [(B^{\#\sim})^{\sim} \vee (C^{\#\sim})^{\sim}]$

$= A^{\#}C^{\#\sim}(B^{\#})^{\sim}Y^{\#}Z^{\#} \vee A^{\#\sim}C^{\#}(B^{\#\sim})^{\sim}X^{\#}Z^{\#} \vee A^{\#\sim}C^{\#\sim}(B^{\#\sim})^{\sim}X^{\#}Y^{\#}$, where $X^{\#} = (B^{\#})^{\sim} \vee (C^{\#\sim})^{\sim}$, $Y^{\#} = (B^{\#\sim})^{\sim} \vee (C^{\#})^{\sim}$ and $Z^{\#} = (B^{\#\sim})^{\sim} \vee (C^{\#\sim})^{\sim}$.

$\Rightarrow ((A \wedge C)^{\sim})^{\#}\ ((B\wedge C)^{\sim\#})^{\sim} \in J.$

Similarly $(A \wedge C)^{\#\sim\sim}\ (B\wedge C)^{\#\sim} \in J.$

Therefore $A\wedge C \overset{c}{-} B \wedge C$.

Suppose $A \overset{c}{-} B$ and $C \overset{c}{-} D \Rightarrow A \wedge C \overset{c}{-} B \wedge C$ and $B \wedge C \overset{c}{-} B \wedge D$.

$\Rightarrow A \wedge C \overset{c}{-} B \wedge D.$

Suppose $A \overset{c}{-} B \Rightarrow A^{\#}(B^{\#})^{\sim}, (A^{\#})^{\sim}B^{\#}, A^{\#\sim}(B^{\#\sim})^{\sim}, (A^{\#\sim})^{\sim}B^{\#\sim} \in J.$

$\Rightarrow A^{\#\sim}(B^{\#\sim})^{\sim}, (A^{\#\sim})^{\sim}B^{\#\sim}, (A^{\#\sim\sim})(B^{\#\sim\sim})^{\sim}, (A^{\#\sim\sim})^{\sim}B^{\#\sim\sim} \in J.$

$\Rightarrow A^{\sim} \overset{c}{-} B^{\sim}.$

Secondly we have to show that $A \overset{c}{-} B \Rightarrow (A * C) \overset{c}{-} (B \overset{c}{-} C)$ and $(C * A) \overset{c}{-} (C * B)$ for all $C\in \mathcal{M}$.

$(A * C)^{\#}, (B * C)^{\#\sim} = A^{\#}(B^{\#})^{\sim} \in J.$

Similarly $[(A * C)^{\#}]\ (B* C)^{\#} \in J.$

$[(A * C)^{\#\sim}]\ [(B* C)^{\#\sim}]^{\sim} = A^{\#\sim}C^{\#\sim}[(B^{\#})^{\sim}(C^{\sim})^{\#}]^{\sim}$

$= A^{\#\sim}C^{\#\sim}[B^{\#} \vee (C^{\#\sim})^{\sim}]$

$= (A^{\#})^{\sim}B^{\#}C^{\#\sim} \in J.$

$\Rightarrow [(A * C)^{\#\sim}]\ [(B* C)^{\#\sim}]^{\sim} \in J.$

Similarly $[(A * C)^{\#\sim}]^{\sim} (B * C)^{\#\sim} \in J$.

Therefore $(A * C) \overset{c}{=} (B * C)$.

$(C * A)^{\#} (C * B)^{\#\sim} = C^{\#} C^{\#\sim} = 0 \in J$.

Therefore $(C * A)^{\#} (C * B)^{\#\sim} \in J$.

Similarly $[(C * A)^{\#}]^{\sim} (C * B)^{\#} \in J$.

$(C * A)^{\#\sim} (C * B)^{\#\sim\sim} = (C^{\#})^{\sim} A^{\#\sim} [(C^{\#})^{\sim} B)^{\#\sim}]^{\sim}$

$= (C^{\#})^{\sim} A^{\#\sim} [(C^{\#}) \vee B^{\#\sim\sim}]$

$= A^{\#\sim} B^{\#\sim\sim} C^{\#\sim}$.

Therefore $(C * A)^{\#\sim} (C * B)^{\#\sim\sim} \in J$.

Similarly $(C * A)^{\#\sim\sim} (C * B)^{\#\sim} \in J$.

Therefore $(C * A) \overset{c}{=} (C * B)$.

Suppose that $A \overset{c}{=} B$ and $C \overset{c}{=} D$.

Since $A \overset{c}{=} B \Rightarrow (A * C) \overset{c}{=} (B * C)$

Since $C \overset{c}{=} D \Rightarrow (B * C) \overset{c}{=} (B * D)$

Therefore $A \overset{c}{=} B$ and $C \overset{c}{=} D \Rightarrow (A * C) \overset{c}{=} (B * D)$.

Hence $\overset{c}{=}$ is a congruence relation on $\mathcal{M}$.

2.20Lemma: Let J be M*-ideal of an A*-algebras over the matrices $(\mathcal{M}, \wedge, \vee, (-)^{\sim}, (-)_{\pi}, *, I)$ and $\overset{c}{=}$ be the congruence relation on $\mathcal{M}$. Then $\overset{c}{=}(0) = J$.

Proof: Let J be M*-ideal of an A*-algebras over the matrices and $\overset{c}{=}$ be the congruence relation on $(\mathcal{M}, \wedge, \vee, (-)^{\sim}, (-)_{\pi}, *, I)$.

Claim: To show that $\overset{c}{=}(0) = J$:

Let $A \in J \Rightarrow A^{\#}, A_{\pi} \in J$

$\Rightarrow A_{\pi}, A_{\pi} \vee A^{\#} \in J$.

$\Rightarrow A_{\pi}, (A_{\pi}^{\sim})^{\sim} \in J$.

$\Rightarrow A_{\pi} 0_{\pi}^{\sim}, A_{\pi}^{\sim} 0_{\pi}, A_{\pi}^{\sim\sim} 0_{\pi}^{\sim}, A_{\pi}^{\sim} 0_{\pi}^{\sim\sim} \in J$.

$\Rightarrow A \overset{c}{=} 0$.

Hence $J \subseteq \overset{c}{=}(0)$...............(i).

Suppose $A \in \overset{c}{=}(0)$.

$\Rightarrow A \overset{c}{=} 0$.

$\Rightarrow A_\pi 0_\pi{}^{\sim}, A_\pi{}^{\sim} 0_\pi, A_\pi{}^{\sim\sim} 0_\pi{}^{\sim}, A_\pi{}^{\sim} 0_\pi{}^{\sim\sim} \in J$.

$\Rightarrow A_\pi, (A_\pi{}^{\sim})^{\sim} \in J$.

$\Rightarrow A_\pi, A_\pi \vee A^{\#} \in J$

$\Rightarrow A^{\#}, A_\pi \in J$

$\Rightarrow A \in J$.

Hence $\overset{c}{=}(0) \subseteq J$................(ii)

From (i) and (ii), $\overset{c}{=}(0) = J$.

.

Reference:

[1] Bellman R.,"Introduction to Matrix Analysis" ,second edition,Mc.Graw Hill, New York ,1970.

[2] Birkoff G**:** Lattice theory, American Mathematical Society, Colloquium Publications,Vol.25, New York, 1948.

[3] David W. Lewis , "Matrix Theory" , Allied Publishers Ltd.,Bombay.

[4] Fernando Guzman and Craig C. Squire:" The Algebra of Conditional logic",Algebra Universalise 27(1990), 88-110.

[5] Goult R.J., "Applied Linear Algebra", Ellis Harwood Limited ,New York.

[6] Koteswara Rao.P, "A*-algebras and If-Then –Else structures", Unpublished PhD Thesis, Nagarjuna University (1994), A.P,India.

[7] Koteswara Rao.P and Venkateswara Rao.J , Prime ideals and Congruences in A*-algebras, Southeast Asian Bullitin of Mathematics (2000), Hong Kong.

[8] Manes E.G**.** "The Equational Theory of Disjoint Alternatives", personal communication to Prof. N.V.Subrahmanyam (1989).

[9] Manes E.G*:* "Ada and the Equational Theory of If-Then-Else",Algebra Universalis 30(1993), 373-394.

[10]Srinivasa Rao .T and Koteswara Rao.P, **"**On Pre A*-Algebras",International Journal of Computational Cognition,Vol.8,No.1 . March 2010.

[11]Suguna Rao.K , Koteswara Rao.P,and Dr.V.Amarendra Babu "Clique on A*-algebras",International journal of Innovation in Science and Mathematics , Volume 2, Issue 3, ISSN (Online): 2347–9051(2014).

Authors profile:

Mr. K. Suguna Rao
presently working as a part time lecture in the Dept of commerce and business administration at Acharya Nagarjuna University, Guntur district, A.P. He did M.sc from Acharya Nagarjuna University. Presently he is pursuing Ph.D. as a full time research scholar at Acharya Nagarjuna University, Nagarjuna Nagar-522510, in the field of algebra under the guidance of Prof. P. Koteswara Rao.He appeared more than 5 national conferences and published 5 research papers in the international journals.

Prof. P. Koteswara Rao
was honoured a PhD from Acharya NagarjunaUniversity, A.P., in the area of Algebra, Topology and OR Techniques. He published 10 research papers in the national and International Journals and participated in 10 national/international Conferences. He guided for 20 M.Phil's and nearly 10 PhD's .Presented research papers and participated in national/international Conferences. Totally he is having more than 26 years of teaching experience. Presently he is working with the capacity of Professor in the dept of commerce and business administration at Acharya Nagarjuna University, NagarjunaNagar-522 510.