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# Strong Efficient Domination in Graphs

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**Abstract**

Let G = (V, E) be a simple graph. E. Sampathkumar and L.Pushpalatha introduced the concepts of strong and weak domination in graphs [5]. In this paper, this concept is extended to efficient domination. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ ($|N_w[v] \cap S| = 1$), where $N_s(v) = \{ u \in V(G): uv \in E(G), deg(u) \geq deg(v) \}$ and $N_w(v) = \{ u \in V(G) : uv \in E(G), deg(v) \geq deg(u)\}$, $N_s[v] = N_s(v) \cup \{v\}$, $N_w[v] = N_w(v) \cup \{v\}$. The minimum cardinality of a strong (weak) efficient dominating set is called strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph is strong efficient if there exists a strong efficient dominating set. In this paper we find classes of graphs which are strong efficient and compare strong efficient domination number with $\gamma_s(G)$, $\Gamma_s(G)$, $i_s(G)$ and $\beta_s(G)$.

**Keywords:** *Strong efficient domination number, Full degree vertex, Strong and weak neighbours.*

**AMS Subject Classification (2010):** 05C69

## 1 Introduction

Throughout this paper, we consider finite, undirected, simple graphs. Let G = (V,E) be a graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by deg u. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex of degree 0 in G is called an isolated vertex and a vertex of degree 1 is called a pendant vertex. For all graph theoretic terminologies and notations, we follow Harary [4]. The following definitions are necessary for the present study.

**1.1 Definition [3]:** A subset M of E(G) is called a matching in G if its elements are edges and no two are adjacent in G; the two ends of an edge in M are said to be matched under M. A matching M saturates a vertex v and v is said to be M-saturated, if some edge of M is incident with v; otherwise v is M-unsaturated. If every vertex of G is M-saturated, then the matching M is called perfect.

**1.2 Definition[2]:** Generalized Hajos graph, denoted by $[K_n]$ having n + $\binom{n}{2}$ vertices is formed by taking $K_n$, adding $\binom{n}{2}$ new vertices and joining each one of them to the ends of exactly one edge of $K_n$.

**1.3 Definition[5] :** A subset S of V(G) of a graph G is called a dominating set if every vertex in V(G) \ S is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

**1.4 Definition[6] :** A subset S of V(G) is called a strong dominating set of G if for every $v \in V - S$ there exists $u \in S$ such that u and v are adjacent and deg u $\geq$ deg v.

**1.5 Definition[1] :** A subset S of V(G) is called an efficient dominating set of G if for every $v \in V(G)$, $|N[v] \cap S| = 1$.

A set of points is independent if no two points are adjacent. A strong dominating set of G which is also independent is called an independent strong dominating set of G. The minimum cardinality of the independent strong dominating set of G is called an independent strong domination number of G and is denoted by $i_s(G)$. The maximum cardinality of an independent strong dominating set of G is called the upper independent strong domination number of G and is denoted by $\beta_s(G)$. Motivated by these definitions, we have defined a new type of domination called strong efficient domination. A graph G is called a strong efficient graph if it admits a strong efficient dominating set. The minimum cardinality of a strong efficient dominating set is called the strong efficient

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domination number of G and is denoted by $\gamma_{se}(G)$. In this paper we classify graphs which are strong efficient and we compare strong efficient domination number with $\gamma_s(G)$, $\Gamma_s(G)$, $i_s(G)$ and $\beta_s(G)$. We have proved that there are graphs in which several strong efficient dominating sets of different cardinalities exist which is interesting in the sense that all efficient dominating sets have the same cardinality $\gamma$.

## 2 Main Results

**Definition 2.1:** Let G = (V, E) be a simple graph. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every v ∈ V(G), $|N_s[v] \cap S| = 1$. ($|N_w[v] \cap S| = 1$), where $N_s(v) = \{ u \in V(G) : uv \in E(G), deg(u) \geq deg(v) \}$ and $N_w(v) = \{ u \in V(G) : uv \in E(G), deg(v) \geq deg(u)\}$, $N_s[v] = N_s(v) \cup \{v\}$ and $N_w[v] = N_w(v) \cup \{v\}$. The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is strong efficient if and only if there exists a strong efficient dominating set of G. Not all graphs admit strong efficient dominating set.

**Example 2.2:** Consider the following figure G.

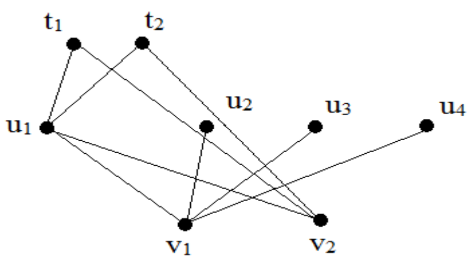


**Figure 1:**

Clearly $\{v_1, v_2\}$ is a strong efficient dominating set of G. Therefore $\gamma_{se}(G) = 2$.

**Example 2.3: Graphs without strong efficient dominating set:**

Consider the graphs $G_1$ and $G_2$ in the following figures 2 and 3 respectively.

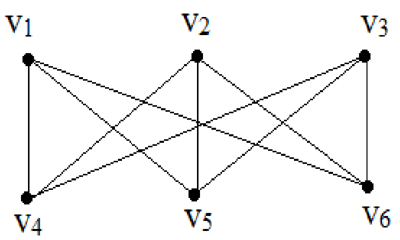


**Figure 2:**

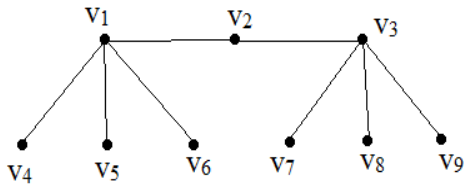


**Figure 3:**

In $G_1$ any dominating set (which is also a strong dominating set) is not efficient. In $G_2$, no strong dominating set is efficient.

**Remark 2.4:** $\gamma_s(G) \leq \gamma_{se}(G)$.

**For,** Let S be a minimum strong efficient dominating set of G.

Let v ∈ V-S. Then $|N_s[v] \cap S| = 1$.

i.e there exists u ∈ S, such that u and v are adjacent and deg u ≥ deg v.

Therefore S is a strong dominating set of G. Therefore $\gamma_s(G) \leq \gamma_{se}(G)$.

**Theorem 2.5:** For any path $P_m$,

$$\gamma_{se}(P_m) = \begin{cases} n \text{ if } m = 3n, n \in N \\ n+1 \text{ if } m = 3n+1, n \in N \\ n+2 \text{ if } m = 3n+2, n \in N \end{cases}$$

**Proof: Case (i):** Let $G = P_{3n}$, n ∈ N. Let $v_1, v_2, v_3 \ldots v_{3n}$ be the vertices of V ($P_{3n}$). $\{v_2, v_5, v_8 \ldots v_{3n-1}\}$ is the unique strong dominating set of $P_{3n}$. It is also the strong efficient dominating set of $P_{3n}$. Therefore $\gamma_{se}(G) = n$.

Therefore $\gamma_{se}(P_{3n}) = n$, for all n ∈ N.

**Case (ii):** Let $G = P_{3n+1}$, n ∈ N. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_{3n}, v_{3n+1}\}$.

$S_1 = \{v_2, v_5, v_8 \ldots v_{3n-1}, v_{3n+1}\}$ and $S_2 = \{v_1, v_3, v_6, v_9, \ldots, v_{3n}\}$ are two strong efficient dominating sets of G.

$S_1 = \{v_2, v_5, v_8 \ldots v_{3n-1}\} \cup \{v_{3n+1}\}$. $|S_1| = n + 1$.

$S_2 = \{v_1\} \cup \{v_3, v_6, v_9, \ldots, v_{3n}\}$. $|S_2| = n + 1$.

Therefore $\gamma_{se}(P_{3n+1}) \leq n + 1$, for all n ∈ N. Since $n + 1 = \gamma_s(P_{3n+1}) \leq \gamma_{se}(P_{3n+1})$. We see that $\gamma_{se}(P_{3n+1}) = n + 1$.

**Case (iii):** Let $G = P_{3n+2}$, n ∈ N. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_{3n}, v_{3n+1}, v_{3n+2}\}$.

$S = \{v_1, v_3, v_6, v_9, \ldots, v_{3n}, v_{3n+2}\}$ is a strong efficient dominating set of G.

That is |S| = n + 2.

Therefore $\gamma_{se}(P_{3n+2}) \leq n + 2$, for all n ∈ N. Since $n + 2 = \gamma_s(P_{3n+2}) \leq \gamma_{se}(P_{3n+2})$. We see that $\gamma_{se}(P_{3n+2}) = n + 2$.

**Examples 2.6:** Consider the path $P_9$

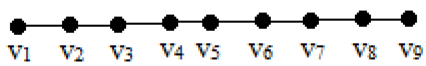


**Figure 4:**

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$\{v_2, v_5, v_8\}$ is the strong efficient dominating set of $P_9$. $\gamma_{se}(P_9) = 3$.

Consider the path $P_{10}$

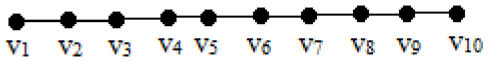


**Figure 5:**

$\{v_1, v_3, v_6, v_9\}$, $\{v_2, v_5, v_8, v_{10}\}$ are two strong efficient dominating sets of $P_{10}$.

$\gamma_{se}(P_{10}) = 4$.

Consider the path $P_{11}$

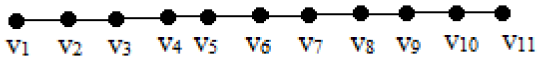


**Figure 6:**

$\{v_1, v_3, v_6, v_9, v_{11}\}$ is the strong efficient dominating set of $P_{11}$. $\gamma_{se}(P_{11}) = 5$.

**Theorem 2.7:** $\gamma_{se}(C_{3n}) = n$, for all $n \in N$.

**Proof:** Let $G = C_{3n}$, $n \in N$. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_{3n}\}$.

$S_1 = \{v_1, v_4, v_7, \ldots, v_{3n-2}\}$, $S_2 = \{v_2, v_5, v_8, \ldots v_{3n-2}\}$ and $S_3 = \{v_3, v_6, v_9, \ldots, v_{3n}\}$ are the strong efficient dominating sets of G.

$|S_1| = |S_2| = |S_3| = n$. $\gamma_{se}(C_{3n}) \le n$. since $n = \gamma_s(C_{3n}) \le \gamma_{se}(C_{3n})$. we see that $\gamma_{se}(C_{3n}) = n$, for all $n \in N$.

**Observations 2.8:**

1. $\gamma_{se}(K_n) = 1$, $n \in N$.
2. $\gamma_{se}(K_{1,n}) = 1$.
3. $\gamma_{se}(W_{n+1}) = 1$, $n \ge 3$, $n \in N$.

**Theorem 2.9:** Every strong efficient dominating set is independent.

**Proof:** Let S be a strong efficient dominating set.

Let u, v ∈ S. Suppose u and v are adjacent.

Let without loss of generality, $d(u) \ge d(v)$.

Then $|N_s[v] \cap S| \ge 2$, a contradiction. Therefore S is independent.

**Theorem 2.10:** If S is a strong efficient dominating set of a connected graph G then $V - S$ is a dominating set of G.

**Proof:** Since every strong efficient dominating set is independent and G is connected, every vertex in S is adjacent to at least one vertex is $V - S$. Therefore $V - S$ is a dominating set of G.

**Remark 2.11:** If S is a strong efficient dominating set of G then $V - S$ need not be a weak dominating set of G.

**Example 2.12:** For the graph G in the following figure

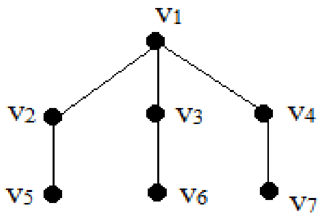


**Figure 7:**

$S = \{v_1, v_5, v_6, v_7\}$ is a strong efficient dominating set of G. Since $v_5, v_6, v_7$ are strongly dominated by the vertices of $V - S$, $V - S$ is not a weak dominating set of G.

**Remark 2.13:**

(i) Every weak efficient dominating set of a graph G is a weak dominating set of G.

(ii) Complement of a strong efficient dominating set of G need not be a weak efficient dominating set of G.(for example, in the graph 2.11, $S = \{v_1, v_5, v_6, v_7\}$ is strong efficient dominating set of G and $V - S$ is not a weak efficient dominating set of G)

**Observation 2.14:** Let G be a connected graph with at least 2 pendant vertices and $\gamma_{se}(G) = 1$. Then G has no perfect matching. (Since $\gamma_{se}(G) = 1$ implies that G has a full degree vertex)

**Theorem 2.15:** Let G be a connected graph and $|V(G)| = n$ (n is even). If $\gamma_{se}(G) > n/2$ then G has no perfect matching

**Proof:** Let S be a $\gamma_{se}$-set of a connected graph G and $|V(G)| = n$.

Let $|S| = t$. Since $\gamma_{se}(G) > \frac{n}{2}$, $t > \frac{n}{2}$.

Suppose that G has a perfect matching M. Then every vertex of G is M-saturated. Therefore every vertex of S is M-saturated. Since S is independent, every line of M has either one end in S and other end in V−S or both ends in V−S. Since M is a perfect matching, $|M| = \frac{n}{2}$. But $|S| = t > \frac{n}{2}$.

Therefore $t - \frac{n}{2}$ vertices of S are M-unsaturated, a contradiction. Therefore G has no perfect matching.

**Remark 2.16:** $i_s(G) \le \gamma_{se}(G) \le \beta_s(G)$.

**Remark 2.17:** There exists graphs in which $\gamma_s(G) < \gamma_{se}(G)$.

**For,** Let $G = D_{r,r}$, $r \ge 2$.

$\gamma_{se}$ (G) = r +1 and $\gamma_s$ (G) = 2.

**Remark 2.18:** Given any positive integer k, there exists a connected graph G such that
$\gamma_{se}$ (G) - $\gamma_s$ (G) = k.

**Example 2.19:** Let G = $D_{k+1,k+1}$, k ≥ 1.
$\gamma_{se}$ (G) = k +2 and $\gamma_s$ (G) = 2.

**Remark 2.20:** It is clear that $\gamma_s$ (G) ≤ $i_s$ (G) ≤ $\gamma_{se}$ (G) ≤ $\beta_s$ (G) ≤ $\Gamma_s$(G).

**Example 2.21:** Consider the following graph G

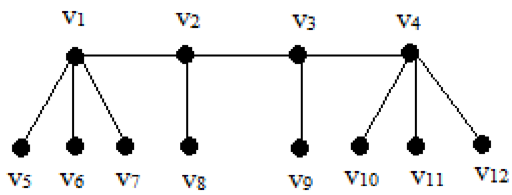


**Figure 8:**

{ $v_1$,$v_2$, $v_3$,$v_4$ } is a strong dominating set of G. Therefore$\gamma_s$(G) = 4.
{ $v_1$,$v_8$, $v_9$,$v_4$ } is an independent strong dominating set of G.
Therefore $i_s$ (G) = 4.
{ $v_1$,$v_8$, $v_9$,$v_4$ } is a strong efficient dominating set of G. Therefore$\gamma_{se}$ (G) = 4.
$\beta_s$ (G) = 4 *and* $\Gamma_s$(G) = 4.
Therefore$\gamma_s$(G) = $i_s$ (G) = $\gamma_{se}$ (G) = $\beta_s$ (G) = $\Gamma_s$(G).

**Example 2.22**

**The following example G shows that strict inequalities occur in the above chain.**

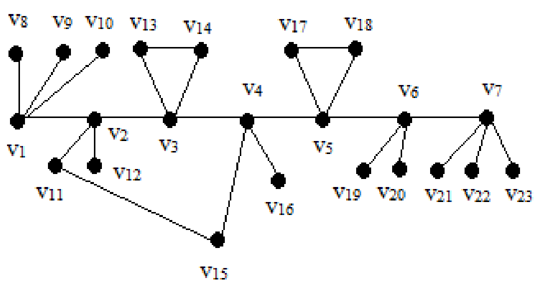


**Figure 9**

$S_0$ = {$v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$} is a strong dominating set of G of minimum cardinality.
Therefore $\gamma_s$(G) = 7.
$S_1$ = {$v_1$, $v_3$, $v_5$, $v_7$, $v_{15}$, $v_{16}$, $v_{19}$, $v_{20}$}, │ $S_1$ │ = 8.
$S_2$ = {$v_1$, $v_3$, $v_6$, $v_{15}$, $v_{16}$, $v_{17}$, $v_{21}$, $v_{22}$, $v_{23}$ },
│ $S_2$ │ = 9.
$S_3$ = {$v_1$, $v_4$, $v_7$, $v_{11}$, $v_{12}$, $v_{13}$, $v_{17}$, $v_{19}$, $v_{20}$ },
│ $S_3$ │ = 9.
$S_4$ = {$v_1$, $v_4$, $v_6$, $v_{11}$, $v_{12}$, $v_{13}$, $v_{17}$, $v_{21}$, $v_{22}$, $v_{23}$ },
│ $S_4$ │ = 10.
$S_5$ = {$v_2$, $v_4$, $v_7$, $v_8$, $v_9$, $v_{10}$, $v_{13}$, $v_{17}$, $v_{19}$, $v_{20}$ },
│ $S_5$ │ = 10.
$S_6$ = { $v_2$, $v_4$, $v_6$, $v_8$, $v_9$, $v_{10}$, $v_{13}$, $v_{17}$, $v_{21}$, $v_{22}$, $v_{23}$ },
│ $S_6$ │ = 11.
$S_7$ = { $v_2$, $v_5$, $v_7$, $v_8$, $v_9$, $v_{10}$, $v_{13}$, $v_{15}$, $v_{16}$, $v_{19}$, $v_{20}$},
│ $S_7$ │=11,
$S_i$, i=1to7 are independent dominating sets of G. $S_1$ is a strong independent dominating set of G of minimum cardinality. Therefore $i_s$ (G) = 8. But $S_1$, $S_2$, $S_4$, $S_5$, $S_6$, $S_7$ are not strong efficient dominating sets of G. $S_3$ is the strong efficient dominating set of G. Therefore $\gamma_{se}$ (G) = 9. $S_8$ = { $v_2$, $v_5$, $v_6$, $v_8$, $v_9$, $v_{10}$, $v_{13}$, $v_{15}$, $v_{16}$, $v_{21}$, $v_{22}$, $v_{23}$}, │ $S_8$ │ = 12. $S_6$ , $S_7$ are minimal strong independent dominating sets of maximum cardinality. Therefore $\beta_s$ (G) = 11. $S_8$ is a minimal strong dominating set of G of maximum cardinality. Therefore $\Gamma_s$(G) = 12. Therefore
$\gamma_s$ (G) < $i_s$ (G) < $\gamma_{se}$ (G) < $\beta_s$ (G) < $\Gamma_s$(G).

**Definition: 2.23[4]:** A regular spanning sub graph of degree 1 is called 1-factor (1F).

**Theorem 2.24**: $K_{n,n}$ −1F is strong efficient and $\gamma_{se}$ ( $K_{n,n}$ −1F ) = 2, ∀ n∈N.

**Proof:** Let G = $K_{n,n}$ −1F.
Let V (G) = {$v_1$, $v_2$,…, $v_n$, $u_1$, $u_2$,…, $u_n$}.
Since we remove 1F from $K_{n,n}$, degree of each vertex is reduced to n− 1.
Each $v_i$ is not adjacent to one $u_j$, ∀ i = 1 to n and ∀ j = 1 to n.
Such {$v_i$, $u_j$} is a strong efficient dominating set.
Hence $\gamma_{se}$ ( $K_{n,n}$ −1F ) = 2, ∀ n∈N.

**Example 2.25:** Consider the following graph $K_{4,4}$ −1F

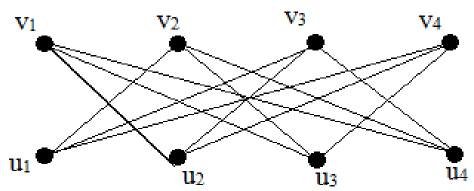


**Figure 10**

{$v_1$,$u_1$}, {$v_2$,$u_2$}, {$v_3$,$u_3$} and {$v_4$,$u_4$} are strong efficient dominating sets.
$\gamma_{se}$ ( $K_{4,4}$ −1F ) = 2.

**Theorem 2.26:** [$K_n$] is strong efficient and $\gamma_{se}$ [$K_n$] = p − Δ ( [$K_n$] ) = $\frac{n^2-3n+4}{2}$ where

$p = |V([K_n])|$.

**Proof:** Let $n \geq 3$. Let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$. Let $G = [K_n]$. $V([K_n]) = \{ v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\binom{n}{2}} \}$. By the definition of $[K_n]$, each $u_i$ is adjacent to exactly 2 vertices of $K_n$.

Therefore $|V([K_n])| = p = n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{n^2+n}{2}$.

$\Delta(G) = \deg v_i$ for any i, i =1 to $n$.

Each $v_i$ is adjacent to the remaining $(n-1)v_i$ s and $(n-1)u_j$ s.

Therefore $\Delta(G) = (n-1) + (n-1) = 2n - 2$.

Total number of $u_j$ s = $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$.

Therefore Number of $u_j$ s which are not adjacent to $v_i = \left(\frac{n^2-n}{2}\right) - (n-1) = \frac{n^2-3n+2}{2}$. These $\frac{n^2-3n+2}{2}$ $u_j$ s together with $v_i$ form a strong efficient dominating set S of G.

Therefore G is strong efficient. $|S| = 1 + \frac{n^2-3n+2}{2} = \frac{n^2-3n+4}{2}$

Therefore $\gamma_{se}(G) \leq \frac{n^2-3n+4}{2}$

Let T be any strong efficient dominating set of $[K_n]$. Since T is independent, T can contain at most one $v_i$, $1 \leq i \leq n$. Since for $n \geq 3$, no $u_j$ can strongly dominate any $v_i$, T contains at least one $v_i$, $(1 \leq i \leq n)$. Therefore T contains exactly one $v_i$. Any $u_j$ can dominate only two $v_i$ $^s$ and all $u_j$ $^s$ are independent. Therefore T contains all $u_j$ $^s$ not adjacent with the $v_i \in T$. therefore $|T| \geq 1 + \left(\frac{n^2-n}{2}\right) - (n-1) = \frac{n^2-3n+4}{2}$. Therefore $\gamma_{se}([K_n]) \geq \frac{n^2-3n+4}{2}$.

Hence $\gamma_{se}([K_n]) = \frac{n^2-3n+4}{2}$.

When n = 2, $[K_n] = C_3$. $\gamma_{se}(C_3) = 1$.

Hence $\gamma_{se}[K_2] = \frac{2^2-3(2)+4}{2}$. Thus $\gamma_{se}([K_n]) = \frac{n^2-3n+4}{2}$

Also $p - \Delta(G) = \frac{n^2+n}{2} - (2n-2) = \frac{n^2+n-4n+4}{2} = \frac{n^2-3n+4}{2}$

$\gamma_{se}(G) = p - \Delta(G)$.

**Example 2.27:** Consider the following graph $G = [K_4]$.

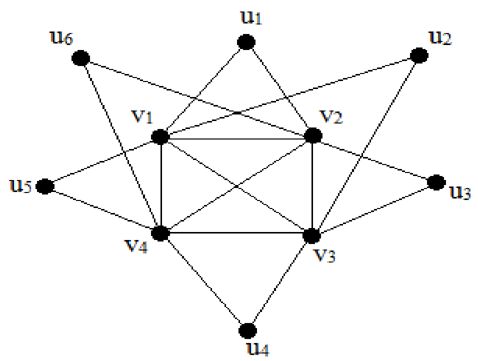


**Figure 11**

$\{ v_1, u_3, u_4, u_6 \}$, $\{ v_2, u_2, u_4, u_5 \}$, $\{ v_3, u_1, u_5, u_6 \}$, $\{ v_4, u_1, u_2, u_3 \}$, are the strong efficient dominating sets of G and $p = 10$, $\Delta(G) = 6$.

Therefore $\gamma_{se}[K_4] = 4 = p - \Delta(G)$.

**Strong efficient dominating sets of different cardinalities in a graph**

In any graph G admitting efficient dominating sets, all efficient dominating sets have the same cardinality namely $\gamma(G)$. This is not true in the case of strong efficient domination. The maximum cardinality of any strong efficient dominating set of G is called the upper strong efficient domination number of G and is denoted by $\Gamma_{se}(G)$. In fact there are graphs in which $\gamma_{se}(G) < \Gamma_{se}(G)$ and for every positive integer k with $\gamma_{se}(G) \leq k \leq \Gamma_{se}(G)$, there are minimal strong efficient dominating sets of different cardinality k . The following examples illustrate this situation.

Let G = (V,E) be a simple graph.

$V(G) = \{v_1, v_2, \ldots v_n, u_1, u_2, \ldots, u_{n+k}, a_1, a_2, \ldots a_{n+k+r}, b_1, b_2, \ldots, b_{n+k+r+s}, \ldots, w_1, w_2, \ldots, w_{n+k+r+s+\ldots+t}, c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_r, \ldots, e_1, e_2, \ldots e_t\}$.

E(G) =

$$\begin{cases} v_1u_i, a_1u_i, b_1u_i, \ldots\ldots\ldots\ldots, w_1u_i, 1 \leq i \leq n+k \\ v_1a_i, u_1a_i, b_1a_i, \ldots\ldots\ldots\ldots, w_1a_i, 1 \leq i \leq n+k+r \\ v_1b_i, u_1b_i, a_1b_i, \ldots\ldots\ldots\ldots, w_1b_i, 1 \leq i \leq n+k+r+s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_1w_i, a_1w_i, b_1w_i, \ldots\ldots\ldots\ldots, 1 \leq i \leq n+k+\ldots+t \\ u_1v_i, a_1v_i, b_1v_i, \ldots\ldots\ldots\ldots, w_1v_i, 1 \leq i \leq n \\ v_2c_i, u_1c_i, a_1c_i, b_1c_i, \ldots\ldots\ldots\ldots, w_1c_i, 1 \leq i \leq k \\ v_2d_i, u_2d_i, a_1d_i, b_1d_i, \ldots\ldots\ldots\ldots, w_1d_i, 1 \leq i \leq r \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_2c_i, u_2c_i, a_2c_i, \ldots\ldots\ldots\ldots, w_1c_i, 1 \leq i \leq t \end{cases}$$

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deg $v_1$ = deg $u_1$ = deg $a_1$ =…. Deg $w_1$ = (n+k) + (n+k+r) + (n+k+r+s) + ….+ (n+k+r+s + …+t) = $\Delta$ (G).

Therefore any strong efficient dominating set of G must contain only one of the vertices $v_1$, $u_1$, $a_1$, $b_1$, $c_1$, ……$w_1$.The sets

$S_1 = \{ v_1, v_2, \ldots v_n \}$. $S_2 = \{ u_1, u_2, \ldots, u_{n+k} \}$

$S_3 = \{ a_1, a_2, \ldots a_{n+k+r} \}$…………………………

$S_n = \{ w_1, w_2, \ldots, w_{n+k+r+s+\ldots+t} \}$ are strong efficient dominating sets of G.

$|S_1|$=n. $|S_2|$=n+k. $|S_3|$=n+k+r, ,………………… $|S_n|$ = n+k+r+s + …+t.

Therefore $\gamma_{se}$ (G) = n and $\Gamma_{se}$ (G) = n+k+r+s + …+t.

**Illustration 2.28:** Let G = (V,E) be a simple graph.

V(G) = {$v_1$, $v_2$, $u_1$, $u_2$, $u_3$, $w_1$, $w_2$, $w_3$, $w_4$, $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $a_1$, $b_1$, $c_1$}.

E(G) =

$$\begin{cases} v_1u_i, w_1u_i, x_1u_i 1 \le i \le 3 \\ v_1w_i, u_1w_i, x_1w_i, 1 \le i \le 4 \\ v_1x_i, u_1x_i, w_1x_i, 1 \le i \le 5 \\ a_1v_2, a_1u_1, a_1w_1, a_1x_1, b_1v_2, b_1u_2, b_1w_1, b_1x_1, c_1v_2, c_1u, c_1w_2, c_1x \end{cases}$$

deg $v_1$ = deg $u_1$ = deg $w_1$ = deg $x_1$ = 12 = $\Delta$ (G).

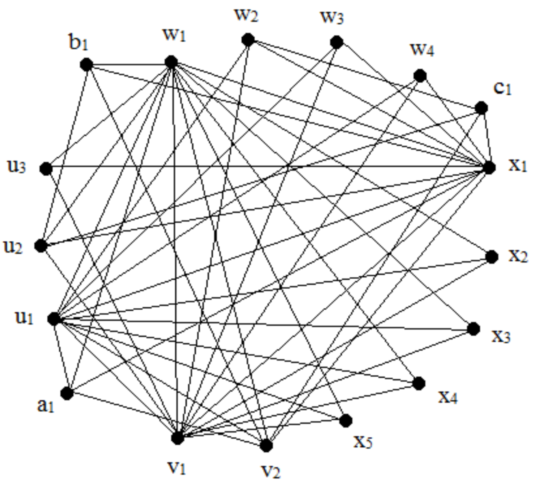


**Figure 12**:

$S_1 = \{ v_1, v_2 \}$, $S_2 = \{ u_1, u_2, u_3 \}$, $S_3 = \{w_1, w_2, w_3, w_4,\}$, and $S_4 = \{ x_1, x_2, x_3, x_4, x_5,\}$are different strong efficient dominating sets of different cardinalities. Therefore $\gamma_{se}$ (G) = 2 and $\Gamma_{se}$(G) = 5.

**Further areas of study**:

**Characterization of strong efficient dominating sets.**

(i) Necessary and sufficient condition for existence of strong efficient dominating sets.

(ii) Strong efficient graphs in which every vertex is contained in a minimum strong efficient dominating set.

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