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Caccioppoli Type Fixed Point Result in Cone b-metric Space

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Abstract
Caccioppoli's fixed point result is extended to the setting of cone b-metric spaces and thus the existing result is generalized.
Keywords: b-metric space; cone b-metric space; fixed point

1.Introduction

Fixed point theory plays a major role in applications of many branches of mathematics. In 1922, Polish mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle [1]. In 1989, Bakhtin introduced b-metric spaces as a generalization of metric spaces [2]. He proved the contraction mapping principle in b-metric spaces to generalize the famous Banach's contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [3–5] and the references therein). In recent investigations, the fixed point in nonconvex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [6–8, 10]). In 2007, Huang and Zhang introduced cone metric spaces as a generalization of metric spaces, also they proved some fixed point theorems for contractive mappings that expanded certain existing results of fixed points in metric spaces. In 2011, Hussain and Shah introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces [10]. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. Throughout this paper, I obtained some fixed point theorems of contractive mappings without the assumption of normality in cone b-metric spaces.

Consistent with Huang and Zhang [6], the following definitions and results will be needed in the sequel.

Let E be a real Banach space and P be a subset of E. Let us denote the zero element of E by θ and the interior of P by int P. The subset P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;

(ii) $a, b \in R$, $a, b \geq 0$; $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

On this basis, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$. Let us write $\| . \|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number satisfying the above is called the normal constant of P. It is well known that $K \geq 1$.

In the followings, we always suppose that E is a Banach space, P is a cone in E with int $P \neq \theta$ and $\leq$ is a partial ordering with respect to P.

Definition 1.1 ([6]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2 ([10]) Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $\theta < d(x, y)$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a cone b-metric space.

Remark 1.3 The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

In [9] , Huang has represented a number of examples, which show that there exist cone b-metric spaces which are not cone metric spaces.

Example 1.4 ([9]) Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \in E$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space, but not a cone metric space.

Example 1.5 ([9]) Let X = {1, 2, 3, 4 }, $E = R^2$,
P = {(x, y) ∈ E : x ≥ 0, y ≥ 0}. Let us define
d : X×X→E by

$$d(x, y) = (|x-y|^{-1}, |x-y|^{-1}), \text{ if } x \neq y,$$
$$= \theta, \text{ if } x = y.$$

Then (X, d) is a cone b-metric space with the coefficient s = 6/5 ; but it is not a cone metric space since the triangle inequality is not satisfied, as follows
d(1, 2) > d(1, 4) + d(4, 2),
d(3,4) > d(3, 1) + d(1, 4).

Definition 1.6 ([10]) Let (X, d) be a cone b-metric space, x ∈ X and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever, for every c ∈ E with θ ≪ c, there exists a natural number N such that $d(x_n, x) \ll c$ for all n ≥ N. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as n→∞.

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every c ∈ E with θ << c, there exists a natural number N such that $d(x_n, x_m) \ll c$ for all n,m ≥ N.

(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence in X is convergent.

The following lemmas are often used, in particular when dealing with cone metric spaces in which the cone need not be normal .

Lemma 1.7 ([8]) Let P be a cone and $\{a_n\}$ be a sequence in E. If c ∈ int P and $\theta \leq a_n \to \theta$ as n→∞, then there exists a natural number N such that for all n > N, we have $a_n \ll c$.

Lemma 1.8 ([8]) Let x, y, z ∈ E, if x ≤ y and y ≪ z, then x ≪ z.

Lemma 1.9 ([10]) Let P be a cone and θ ≤ u ≪ c for each c ∈ int P, then u = θ .

Lemma 1.10 ([11]) Let P be a cone. If u ∈ P and u ≤ ku for some 0 ≤ k < 1, then u = θ .

Lemma 1.11 ([8]) Let P be a cone and a ≤ b + c for each c ∈ int P, then a ≤ b.

2. Main Result

In this section, we will present some fixed point theorems for contractive mappings in the setting of cone b-metric spaces. Furthermore, we will give examples to support our main results.

Theorem 2.1 Let (X,d) be a complete b-cone metric space with coefficient s ≥ 1. Let T : X → X be a self mapping such that for each positive integer 'n' ,
$d(T^n x, T^n y) \leq \alpha_n d(x, y)$, for all x, y ∈ X , where $\alpha_n > 0$ ∀ n and are independent of x and y.
If the series $\sum_{n=1}^{\infty} \alpha_n$ is convergent, then T has a unique fixed point in X.

Proof Let x_0 be an arbitrary element of X , then $Tx_0 \in X$ and let $Tx_0 = x_1$.
Again since $x_1 \in X$; $Tx_1 \in X$ and let $Tx_1 = x_2$.
Proceeding in this way, we will get a sequence $\{x_n\}$ in X, such that
$Tx_n = x_{n+1}$, ∀n.
Also, $x_n = Tx_{n-1} = T(Tx_{n-2}) = T^2 x_{n-2} = \cdots\cdots = T^{n-1} x_1 = T^n x_0$; ∀n
Let n and p be any two positive integers, then

$$d(x_{n+p}, x_n)$$
$$\leq s\, d(x_{n+p}, x_{n+p-1}) + s\, d(x_{n+p-1}, x_n)$$
$$\leq s\, d(x_{n+p}, x_{n+p-1}) + s^2 d(x_{n+p-1}, x_{n+p-2}) + s^2 d(x_{n+p-2}, x_n)$$
$$\leq s\, d(x_{n+p}, x_{n+p-1}) + s^2 d(x_{n+p-1}, x_{n+p-2}) + s^3 d(x_{n+p-2}, x_{n+p-3}) + s^3 d(x_{n+p-3}, x_n)$$
$$\leq \cdots\cdots$$
$$\leq s d(x_{n+p}, x_{n+p-1}) + s^2 d(x_{n+p-1}, x_{n+p-2}) + s^3 d(x_{n+p-2}, x_{n+p-3}) + \ldots + s^{p-1} d(x_{n+2}, x_{n+1}) + s^{p-1} d(x_{n+1}, x_n)$$
$$= s\, d(T^{n+p-1}x_1, T^{n+p-1}x_0) + s^2 d(T^{n+p-2}x_1, T^{n+p-2}x_0) + s^3 d(T^{n+p-3}x_1, T^{n+p-3}x_0) + \ldots + s^{p-1} d(T^{n+1}x_1, T^{n+1}x_0) + s^{p-1} d(T^n x_1, T^n x_0)$$
$$\leq s\, \alpha_{n+p-1} d(x_1, x_0) + s^2 \alpha_{n+p-2} d(x_1, x_0) + s^3 \alpha_{n+p-3} d(x_1, x_0) + \ldots + s^{p-1} \alpha_{n+1} d(x_1, x_0) + s^{p-1} \alpha_n d(x_1, x_0)$$
$$\leq s^p [\alpha_{n+p-1} d(x_1, x_0) + \alpha_{n+p-2} d(x_1, x_0) + \alpha_{n+p-3} d(x_1, x_0) + \ldots + \alpha_{n+1} d(x_1, x_0) + \alpha_n d(x_1, x_0)]$$
$$= s^p d(x_1, x_0) [\alpha_{n+p-1} + \alpha_{n+p-2} + \alpha_{n+p-3} + \ldots + \alpha_{n+1} + \alpha_n]$$
$$= s^p d(x_1, x_0) \sum_{k=n}^{n+p-1} \alpha_k$$

Let c ∈ P be chosen arbitrarily, such that θ ≪ c. Then there exists a real number δ > 0, such that ∀ x∈P with ‖ x ‖ < δ; x≪c .
If $x_0 = x_1$, then x_0 would have been a fixed point for T. Let $x_0 \neq x_1$ and so $d(x_0, x_1) \neq 0$.
Let $\mu = \frac{\delta}{2 s^p \| d(x_1, x_0) \|}$

Now, $\sum_{n=1}^{\infty} \alpha_n$ is convergent and so for this μ, there exists a positive integer N such that for all n > N and for all positive integral values of p,
$\sum_{k=n}^{n+p-1} \alpha_n < \mu$

Thus,for all n > N and for all p,

$$d(x_{n+p}, x_n)$$
$$\leq s^p d(x_1, x_0) \sum_{k=n}^{n+p-1} \alpha_n$$
$$< s^p d(x_1, x_0)\, \mu$$
$$= s^p d(x_1, x_0) \frac{\delta}{2 s^p \| d(x_1, x_0) \|}$$

$= d(x_1, x_0) \frac{\delta}{2\,||\,d(x_1, x_0)\,||}$

Thus, $\|d(x_{n+p}, x_n)\| < \delta/2 < \delta$

$\Rightarrow$ $d(x_{n+p}, x_n) \ll c$, $\forall n > N$ and for all positive integral values of p.

Thus, $\{x_n\}$ is a Cauchy sequence in X and X is complete.

So, $x_n \to \xi \in X$ (say) as $n \to \infty$.

Now, $d(\xi, T\xi) \leq s\, d(\xi, x_n) + s\, d(x_n, T\xi)$; n be any positive integer.

$= s\, d(x_n, \xi) + s\, d(Tx_{n-1}, T\xi)$

$\leq s\, d(x_n, \xi) + s\, \alpha_1 d(x_{n-1}, \xi)$

$\to \theta$ as $n \to \infty$

Hence, $\xi = T\xi$, which implies that ξ is a fixed point of T in X.

To Prove the uniqueness of ξ, let us assume there exists another fixed point ζ of T in X.

Then $\zeta = T\zeta = T(T\zeta) = T^2\zeta = \ldots\ldots = T^n\zeta$; $\forall n$ and similarly $\xi = T^n\xi$; $\forall n$. Then

$d(\zeta, \xi) = d(T^n\zeta, T^n\xi) \leq \alpha_n d(\zeta, \xi)$, for all positive integral values of n

Now, if $d(\zeta, \xi) \neq \theta$ then $\alpha_n \geq 1$ $\forall n$.

This means α_n does not tend to 0 as $n \to \infty$ and hence $\sum_{n=1}^{\infty} \alpha_n$ is not convergent.

So, $d(\zeta, \xi) = \theta \Rightarrow \zeta = \xi$

which proves the uniqueness of the fixed point of T in X.

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