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On Cyclic Orthogonal Double Covers of Circulant Graphs by Certain Graphs

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Abstract

An orthogonal double cover (ODC) of a graph H is a collection $\mathcal{G} = \{G_v : v \in V(H)\}$ of $|V(H)|$ subgraphs (pages) of H, such that they cover every edge of H twice and the intersection of any two of them contains exactly one edge. An ODC $\mathcal{G}$ of H is cyclic (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$. In this paper we are concerned with CODC of circulant graphs by a special class of trees and a special class of connected graphs.

Keywords

Graph decomposition; Cyclic orthogonal double cover; Automorphism group; Orthogonal- labelling.

1. Introduction

All graphs we deal with are undirected, finite and simple. Let H be any regular graph, and let $\mathcal{G} = \{G_0, G_1, \cdots, G_{|V(H)|-1}\}$ be a collection of $|V(H)|$ subgraphs (pages) of H. The collection $\mathcal{G}$ is an orthogonal double cover (ODC) of H if **(i)** Every edge of H is contained in exactly two of the pages in $\mathcal{G}$, and **(ii)** For any two distinct pages G_i and $G_j \in \mathcal{G}$, $|E(G_i) \cap (G_j)| = 1$, if and only if i and j are adjacent in H.

If all pages are isomorphic to a given graph G, then $\mathcal{G}$ is an ODC of H by G. According to the obvious properties of ODCs by a graph G, the underlying graph H has to be $|E(G)|$-regular. This concept is a generalization of the definitions of an ODC of complete graphs and complete bipartite graphs, which has been studied extensively [1]- [2]. El-Shanawny et al. studied extensively the ODC of complete bipartite graphs; see [3, 4, 5, 6]. An effective method to construct ODCs in the above cases was based on the idea of translate a given subgraph G by a group acting on $V(H)$. If the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$ (the set of all automorphisms of $\mathcal{G}$), then an ODC $\mathcal{G}$ of H is cyclic (CODC). Therefore, the circulant graph is of special interest.

For a sequence $\{d_1, d_2, \ldots, d_k\}$ of positive integers with $1 \le d_1 \le d_2 \le \cdots \le d_k \le \lfloor n/2 \rfloor$, the circulant graph $Circ(n; \{d_1, d_2, \ldots, d_k\})$, has vertex set $\mathbb{Z}_n = \{0,1, \ldots, n-1\}$; two vertices v_1 and v_2 are adjacent, if and only if $v_1 - v_2 \equiv \pm d_i \ (mod\ n)$, for some i, $i \in \{1,2, \cdots, k\}$.

For an edge $\{v_1, v_2\}$ in $Circ(n; \{d_1, d_2, \ldots, d_k\})$, the length of $\{v_1, v_2\}$ is $min\,\{|v_1 - v_2|, n - |v_1 - v_2|\}$. Given two edges $e_1 = \{u_0, u_1\}$ and $e_2 = \{v_0, v_1\}$ of the same length l in $Circ(n; \{d_1, d_2, \ldots, d_k\})$, the rotation distance $r(l)$ between e_1 and e_2 is $r(l) = min\{r_1, r_2 : (u_0 + r_1)(u_1 + r_1) = e_2,\ (v_0 + r_2)(v_1 + r_2) = e_1\}$, where addition and difference are calculated inside $\mathbb{Z}_n$. Note that if $r(l) = l$, then the edges e_1 and e_2 are adjacent; if $r(l) \ne l$, then the edges e_1 and e_2 are non adjacent.

Throughout the paper we make use of the usual notation: K_n for the complete graph on n vertices, $K_{m,n}$ for the complete bipartite graph with independent sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle on n vertices, $D \cup F$ for the disjoint union of D and F, $D \cup^v F$ for the union of D and F with a common vertex v belongs to F and D, and $D \cup^{\{a,b\}} F$ for the union of D and F with a common edge $\{a, b\}$ belongs to F and D. Other terminology not defined here can be found in [8]. The paper is organized as follows: section 1.1, describes the technique that can be used throughout, section 2, constructs a CODC circulant graphs by a special class of trees and section 3, constructs a CODC circulant graphs by a certain connected graphs.

1.1. CODC of circulant graphs

Consider the complete graph $K_n = Circ(n; \{1,2, \ldots, \lfloor n/2 \rfloor\})$. The authors of [9] introduced the notion of an orthogonal labelling. Given a graph $G = (V, E)$ with $n - 1$ edges, a $1-1$ mapping $\psi: V \to \mathbb{Z}_n$ is an orthogonal labelling of G if the following conditions are satisfied:

i. For every $l \in \{1,2, \ldots, \lfloor (n-1)/2 \rfloor\}$, G contains exactly two edges of length l, and exactly one edge of length $(n/2)$ if n is even, and

ii. For every $l \in \{1,2, \ldots, \lfloor (n-1)/2 \rfloor\}$, $r(l) = \{1,2, \ldots, \lfloor (n-1)/2 \rfloor\}$.

The following theorem of Gronau et al. [9] relates CODCs of K_n and orthogonal labellings.

Theorem 2. ([9]) *A CODC of K_n by a graph G exists if and only if there exists an orthogonal 2abeling of G.*

Sampathkumar and Srinivasan [7], called an orthogonal $\{1,2, \ldots, \lfloor n/2 \rfloor\}$-labelling and generalized it to an orthogonal $\{d_1, d_2, \ldots, d_k\}$-labelling, where $\{d_1, d_2, \ldots, d_k\}$ is a sequence of positive integers with $1 \le d_1 \le d_2 \le \cdots \le d_k\} \le \lfloor n/2 \rfloor$.

(i) Either n is odd or even and $d_k \ne n/2$:

Given a subgraph G of $Circ(n; \{d_1, d_2, \ldots, d_k\})$ with $2k$ edges, a labelling of G, in $\mathbb{Z}_n$, is an orthogonal $\{d_1, d_2, \ldots, d_k\}$–labelling of G if:

a) For every $l \in \{d_1, d_2, \ldots, d_k\}$, G contains exactly two edges of length l, and

b) $\{r(l) : l \in \{d_1, d_2, \ldots, d_k\} = \{d_1, d_2, \ldots, d_k\}$.

(ii) n is even and $d_k = n/2$

Given a subgraph G of $Circ(n;\{d_1, d_2, \dots, d_{k-1}, n/2\})$ with $2k-1$ edges, a labelling of G, in $\mathbb{Z}_n$, is an orthogonal $\{d_1, d_2, \dots, d_{k-1}, n/2\}$-labelling of G if:

a) For every $l \in \{d_1, d_2, \dots, d_{k-1}\}$, G contains exactly two edges of length l, and G contains exactly one edges of length $(n/2)$, and

b) $\{r(l): l \in \{d_1, d_2, \dots, d_{k-1}\}, = \{d_1, d_2, \dots, d_{k-1}\}$, The following theorem of Sampathkumar and Simaringa [7], is a generalization of Theorem 2.

Theorem 3 ([7]) *A CODC of $Circ(n;\{d_1, d_2, \dots, d_k\})$ by a graph G exists, if and only if there exists an orthogonal $\{d_1, d_2, \dots, d_k\}$-labelling of G.*

2. CODCs of circulant graphs by tree $T(w, w_1, w_2)$

Let $w \geq 1$ be the number of edges of the path $P_{w+1} =: x_0 x_1 x_2 \cdots x_{w-1} x_w$ and $w_1,\ w_2\ \geq 1$ be the numbers of edges apart from the path P_{w+1} that all attached to the vertex x_0 and x_w respectively. This tree will be denoted by $\boldsymbol{T(w, w_1, w_2)}$ as in figure 1. In the following theorem we prove the existence of CODCs of circulant graph by the tree for $1 \leq w \leq 5$.

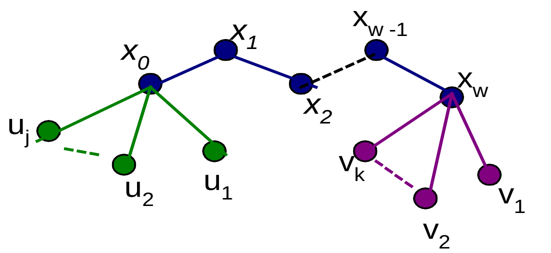

Figure 1 $T(w, w_1, w_2)$ with $w_1 = j$ and $w_2 = k$

Theorem 4. *Let n, l, w, w_1 and w_2 be positive integers, then for $1 \leq w \leq 5$ there is a CODC of $cir(n;\ D)$ by $T(w, w_1, w_2)$, where $D \subseteq \left\{l : 1 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor\right\}$.*

Proof. For each case of w we will define the vertices of $T(w, w_1, w_2)$ that construct orthogonal- D labeling of $T(w, w_1, w_2)$.

Case 1. $w = 1$, $n \geq 8$, $w_1 = 3$, $w_2 = n - 7$ and $D = \left\{l :\ 1 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor : l \neq 3\right\}$, the vertices of $T(w, w_1, w_2)$ are labelled by

$$\psi(x_i) = i + 1 :\ 0 \leq i \leq 1,$$

$$\psi(u_j) = \begin{cases} 0, & if \quad j = 1; \\ 3, & if \quad j = 2; \\ n-1, & if \quad j = 3, \end{cases}$$

$$\psi(v_K) =\ n - 1 - k:\ \ 1 \leq k \leq n - 7.$$

Then the edges of length 1 are $\{\psi(u_1),\psi(x_0)\}$ and $\{\psi(x_0),\psi(x_1)\}$; those of length 2 are $\{\psi(x_0),\psi(u_2)\}$ and $\{\psi(u_3),\psi(x_0)\}$; those of length l where $4 \le l \le \lfloor n/2 \rfloor$ are $\{\psi(x_1),\psi(v_{n-l-3})\}$ and $\{\psi(v_{l-3}),\psi(x_1)\}$.

Case 2 $w = 2$

Subcase 2.1. $n = 2m,\ m \ge 5\ w_1 = 2,\ w_2 = 2m-9$ and $D = \{\, l : 1 \le l \le m \text{ and } l \notin \{2,3\}\}$, the vertices of $T(w,w_1,w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} m+2, & if \quad i = 0; \\ 1, & if \quad i = 1; \\ m, & if \quad i = 2, \end{cases}$$

$$\psi(u_j) = \begin{cases} m+3, & if \quad j = 1; \\ m+1, & if \quad j = 2, \end{cases}$$

$$\psi(v_K) = m+3+k : \quad 1 \le k \le 2m-7 : \ k \notin \{m-4, m-2\}.$$

Then the edges of length 1 are $\{\psi(x_0),\psi(u_1)\}$ and $\{\psi(u_2),\psi(x_0)\}$; those of length $m-1$ are $\{\psi(x_1),\psi(x_2)\}$ and $\{\psi(x_0),\psi(x_1)\}$; those of length l where $\{4 \le l \le m : \ l \ne m-1\}$ are $\{\psi(x_2),\psi(v_{l-3})\}$ and $\{\psi(v_{n-l-3}),\psi(x_2)\}$.

Subcase 2.2 $n = 2m+1,\ m \ge 5\ w_1 = 2,\ w_2 = 2m-8$ and $D = \{l : 1 \le l \le m \text{ and } l \ne 3\}\}$, the vertices of $T(w,w_1,w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} m+2, & if \quad i = 0; \\ 1, & if \quad i = 1; \\ m+1, & if \quad i = 2, \end{cases}$$

$$\psi(u_j) = \begin{cases} m+4, & if \quad j = 1; \\ m, & if \quad j = 2, \end{cases}$$

$$\psi(v_K) = m+4+k : \quad 1 \le k \le 2m-6 : \ k \notin \{m-3, m-2\}.$$

Then the edges of length 2 are $\{\psi(x_0),\psi(u_1)\}$ and $\{\psi(u_2),\psi(x_0)\}$; those of length m are $\{\psi(x_1),\psi(x_2)\}$ and $\{\psi(x_0),\psi(x_1)\}$; those of length l where $4 \le l \le m-1$ are $\{\psi(x_2),\psi(v_{l-3})\}$ and $\{\psi(v_{n-l-3}),\psi(x_2)\}$.

Case 3 $w = 3$

Subcase 3.1. $n = 2m,\ m > 3\ w_1 = 2,\ w_2 = 2m-7$ and $D = \{l : 1 \le l \le m-1\}$, the vertices of $T(w,w_1,w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} 1, & if \quad i = 0; \\ m, & if \quad i = 1; \\ 2m-1, & if \quad i = 2; \\ m+1, & if \quad i = 3, \end{cases}$$

$$\psi(u_j) = \begin{cases} 0, & if \quad j = 1; \\ 2, & if \quad j = 2, \end{cases}$$

$\psi(v_K) = m+2+k: \quad 1 \le k \le 2m-3: \ k \notin \{m-3, m-2, m-1, m\}$.

Then the edges of length 1 are $\{\psi(u_1), \psi(x_0)\}$ and $\{\psi(x_0), \psi(u_2)\}$; those of length $m-1$ are $\{\psi(x_0), \psi(x_1)\}$ and $\{\psi(x_1), \psi(x_2)\}$; those of length l where $2 \le l \le m-2$ are $\{\psi(x_3), \psi(v_{l-1})\}$ and $\{\psi(v_{n-l-1}), \psi(x_3)\}$.

Subcase 3.2. $n = 2m+1,\ m > 3\ \ w_1 = 2,\ \ w_2 = 2m-7$ and $D = \{l : 1 \le l \le m-1\}$, the vertices of $T(w, w_1, w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} 1, & if \quad i = 0; \\ m, & if \quad i = 1; \\ 2m-1, & if \quad i = 2; \\ m+1, & if \quad i = 3, \end{cases}$$

$$\psi(u_j) = \begin{cases} 0, & if \quad j = 1; \\ 2, & if \quad j = 2, \end{cases}$$

$\psi(v_K) = m+2+k: \quad 1 \le k \le 2m-2: \ k \notin \{m-3, m-2, m-1, m, m+1\}$.

Then the edges of length 1 are $\{\psi(u_1), \psi(x_0)\}$ and $\{\psi(x_0), \psi(u_2)\}$; those of length $m-1$ are $\{\psi(x_0), \psi(x_1)\}$ and $\{\psi(x_1), \psi(x_2)\}$; those of length l where $2 \le l \le m-2$ are $\{\psi(x_3), \psi(v_{l-1})\}$ and $\{\psi(v_{n-l-1}), \psi(x_3)\}$.

Case 4. $w = 4$

Subcase 4.1. $n = 2m,\ m > 4\ \ w_1 = 1,\ \ w_2 = 2m-8$ and $D = \{\ l :\ 1 \le l \le m\ and\ l \ne m-2\}$, the vertices of $T(w, w_1, w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} m, & if \quad i = 0; \\ 1, & if \quad i = 1; \\ 2, & if \quad i = 2; \\ m+2, & if \quad i = 3; \\ 0, & if \quad i = 4, \end{cases}$$

$\psi(u_1) = m-2$,

$\psi(v_K) = k+2: \quad 1 \le k \le 2m-5: \ k \notin \{m-4, m-2, m\}$.

Then the edges of length 2 are $\{\psi(x_4), \psi(x_1)\}$ and $\{\psi(x_1), \psi(x_2)\}$; those of length 2 are $\{\psi(x_0), \psi(x_3)\}$ and $\{\psi(u_1), \psi(x_0)\}$; the edge of length m are $\{\psi(x_2), \psi(x_3)\}$; those of length l where $\{3 \le l \le m-1 : l \ne m-2\}$ are $\{\psi(x_4), \psi(v_{l-2})\}$ and $\{\psi(v_{n-l-2}), \psi(x_4)\}$.

Subcase 4.2. $n = 2m+1,\ m > 3\ \ w_1 = 1,\ \ w_2 = 2m-7$ and $D = \{\ l: 2 \le l \le m\}$, the vertices of $T(w, w_1, w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} m+2, & if \quad i = 0; \\ 1, & if \quad i = 1; \\ m, & if \quad i = 2; \\ 2m-1, & if \quad i = 3; \\ m+1, & if \quad i = 4, \end{cases}$$

$\psi(u_1) = 2,$

$\psi(v_K) = m + k + 2 : \quad 1 \le k \le 2m - 2: \; k \notin \{m-2, m-1, m, m+1\}.$

Then the edges of length $m-1$ are $\{\psi(x_1), \psi(x_2)\}$ and $\{\psi(x_2), \psi(x_3)\}$; those of length m are $\{\psi(u_1), \psi(x_0)\}$ and $\{\psi(x_0), \psi(x_1)\}$; those of length l where $2 \le l \le m-2$ are $\{\psi(x_4), \psi(v_{l-1})\}$ and $\{\psi(v_{n-l-2}), \psi(x_4)\}$.

Case 5. $w = 5, n > 10, \; w_1 = 1, \; w_2 = n - 9$ and $D = \left\{l: 1 \le l \le \left\lfloor \frac{n}{2} \right\rfloor \; and \; l \ne 4\right\}$, the vertices of $T(w, w_1, w_2)$ are labelled by

$$\psi(x_i) = \begin{cases} 5, & if \quad i = 0; \\ 4, & if \quad i = 1; \\ 1, & if \quad i = 2; \\ n-2, & if \quad i = 3; \\ 0, & if \quad i = 4; \\ 2, & if \quad i = 5, \end{cases}$$

$\psi(u_1) = 6,$

$\psi(v_K) = n - 2 - k : \quad 1 \le k \le n - 9.$

Then the edges of length 1 are $\{\psi(x_0), \psi(u_1)\}$ and $\{\psi(x_1), \psi(x_0)\}$; those of length 2 are $\{\psi(x_4), \psi(x_5)\}$ and $\{\psi(x_3), \psi(x_4)\}$; those of length 3 are $\{\psi(x_2), \psi(x_1)\}$ and $\{\psi(x_3), \psi(x_2)\}$; those of length l where $5 \le l \le \left\lfloor \frac{n}{2} \right\rfloor$ are $\{\psi(v_{l-4}), \psi(x_5)\}$ and $\{\psi(x_5), \psi(v_{n-l-4})\}$.

3. CODCs of circulant graphs by certain connected graphs

Theorem 5 *For any positive integer $n > 1$, here exists a CODC of $4n$-regular $Circ(4n+2; \{1,2,\dots,2n\})$ by $K_{2,2n-2} \cup^{2n+2} K_{2,2}$.*

proof Let us define $\psi: V\left(K_{2,2n-2} \cup^{2n+2} K_{2,2}\right) \to \mathbb{Z}_{4n+2}$ by

$$\psi(v_i) = \begin{cases} 2n+1+i, & if \quad 0 \le i \le n-1; \\ i+2, & if \quad n \le i \le 2n-2; \\ 1, & if \quad i = 2n-1; \\ 0, & if \quad i = 2n; \\ 3n+2, & if \quad i = 2n+1; \\ 3n+3, & if \quad i = 2n+2. \end{cases}$$

Then the edges of length l where $1 \le l \le n-1$ are $\{\psi(v_0), \psi(v_l)\}$ and $\{\psi(v_0), \psi(v_{2n-l-1})\}$; those of length n are $\{\psi(v_1), \psi(v_{2n+1})\}$ and $\{\psi(v_{2n-1}), \psi(v_{2n+2})\}$; those of length $n+1$ are $\{\psi(v_1), \psi(v_{2n+2})\}$ and $\{\psi(v_{2n-1}), \psi(v_{2n+1})\}$; those of length l where $n+2 \le l \le 2n$ are $\{\psi(v_{2n-l+1}), \psi(v_{2n})\}$ and $\{\psi(v_{2n}), \psi(v_{l-2})\}$. For every $l \in$

$\{1,2,\dots,2n\})$, $K_{2,2n-2} \cup^{2n+2} K_{2,2}$ contains exactly two edges of length l, and since every two edges of the same length are adjacent then $\{r(l) = \{1,2,3,\dots,2n\}: l \in \{1,2,3,\dots,2n\}\}$ and hence $K_{2,2n-2} \cup^{2n+2} K_{2,2}$ has an orthogonal labelling. By Theorem 3, there exists a CODC of $4n$-*regular* $Circ(4n+2;\ \{1,2,\dots,2n\})$ *by* $K_{2,2n-2} \cup^{2n+2} K_{2,2}$ for $n > 1$. □

Theorem 6 *For any positive integer* $n > 7$, *there exists a CODC of* $(n-1)$-*regular* $Circ(n;\ \{1,2,\dots,\lfloor n/2 \rfloor\})$ *by* $\left(C_5 \cup^{\{\alpha,\alpha+4\}} K_3\right) \cup^{\alpha+4} K_{1,n-8}$.

Proof Let us define $\psi: V\left(\left(C_5 \cup^{\{\alpha,\alpha+4\}} K_3\right) \cup^{\alpha+4} K_{1,n-8}\right) \to \mathbb{Z}_n$ by

$$\psi(v_i) = \begin{cases} \alpha, & if \quad i = 0; \\ \alpha + i + 1, & if \quad 1 \le i \le 5; \\ \alpha + i + 2, & if \quad 6 \le i \le n-3, \end{cases}$$

where $\alpha \in \mathbb{Z}_n$.

Then the edges of length 1 are $\{\psi(v_3), \psi(v_4)\}$ and $\{\psi(v_5), \psi(v_4)\}$; those of length 2 are $\{\psi(v_0), \psi(v_1)\}$ and $\{\psi(v_1), \psi(v_3)\}$; ; those of length 3 are $\{\psi(v_0), \psi(v_2)\}$ and $\{\psi(v_2), \psi(v_5)\}$; those of length 4 are $\{\psi(v_0), \psi(v_3)\}$ and $\{\psi(v_3), \psi(v_6)\}$; those of length l where $5 \le l \le \lfloor n/2 \rfloor$ are $\{\psi(v_3), \psi(v_{l+2})\}$ and $\{\psi(v_3), \psi(v_{n-l+2})\}$. For every $l \in \{1,2,\dots,\lfloor n/2 \rfloor\})$, $C_5 \cup^{\alpha+4} K_3 \cup^{\alpha+4} K_{1,n-8}$ contains exactly two edges of length l, and since every two edges of the same length are adjacent then $\{r(l) = \{1,2,3,\dots,\lfloor n/2 \rfloor\}: l \in \{1,2,3,\dots,\lfloor n/2 \rfloor\}\}$ and hence $C_5 \cup^{\alpha+4} K_3 \cup^{\alpha+4} K_{1,n-8}$ has an orthogonal labelling.

By Theorem 3, there exists a CODC of $(n-1)$-*regular* $Circ(n;\ \{1,2,\dots,\lfloor n/2 \rfloor\})$ *by* $\left(C_5 \cup^{\{\alpha,\alpha+4\}} K_3\right) \cup^{\alpha+4} K_{1,n-8}$ for $n > 7$. □

As a special case, if $n = 8$, we construct CODCs of $Circ(n;\ \{1,2,\dots,\lfloor n/2 \rfloor\})$ *by* $\left(C_5 \cup^{\{\alpha,\alpha+4\}} K_3\right)$.

The following theorem constructs *a CODC* of circulant graph by the complete bipartite graph $K_{2,2n}$.

Theorem 7 *For any positive integer* $n > 1$, *there exists a CODC of* $4n$-*regular* $Circ(4n+2;\ \{1,2,\dots,2n\})$ *by* $K_{2,2n}$.

Proof Let us define $\psi: V\left(K_{2,2n}\right) \to \mathbb{Z}_{4n+2}$ by

$$\psi(v_i) = \begin{cases} n + 1 + i, & if \quad 0 \le i \le n-1; \\ n + 2 + i, & if \quad n \le i \le 2n-1; \\ \alpha, & if \quad i = 2n; \\ \alpha + 2n + 1, & if \quad i = 2n+1, \end{cases}$$

where $\alpha \in \mathbb{Z}_{4n+2}$.

Then the edges of length l where $1 \le l \le n$ are $\{\psi(v_{2n+1}), \psi(v_{n-l})\}$ and $\{\psi(v_{2n+1}), \psi(v_{l+n-1})\}$; those of length l where $n+1 \le l \le 2n$ are $\{\psi(v_{2n}), \psi(v_{l-n-1})\}$ and $\{\psi(v_{2n}), \psi(v_{3n-l})\}$. For every $l \in \{1,2,\dots,2n\})$, $K_{2,2n}$ contains exactly two edges of length l, and since every two edges of the same length are adjacent then $\{r(l) = \{1,2,3,\dots,2n\}: l \in \{1,2,3,\dots,2n\}\}$ and hence $K_{2,2n}$ has an orthogonal labelling. By Theorem 3, there exists a CODC of $4n$-*regular* $Circ(4n+2;\ \{1,2,\dots,2n\})$ *by* $K_{2,2n}$ for $n > 1$. □

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