where and are determined by the set of the following equations :

$$+ p = q$$

and

$$\frac{s}{} + \quad = r.$$

Substituting the values of , s and r we see that

$$\frac{s}{} + \quad r = 0:$$

The values of are given by the equation

$$(\ _{i1} + 10 \ _{i} \ _{i}) ^{2} + \ _{i1} (10 \ _{i1} + \ _{i}) \ h$$

$$12 \ _{i} \ _{i1} + 4 \ _{i} \ _{i1} + (12 \ _{i} \ _{i1} + 4 \ _{i} \ _{i1}) ^{2} = 0:$$

Let us consider $= 1 +$; for small positive ; i.e., $< \frac{1}{2}$: The value of the above expression is

$$(12 \ _{i1} (1 \ ) + \ _{i1} ^{2} + 12 \ _{i} (1 \ 2 \ ) + \ _{i} ^{2} (13 \ ))h$$

$$+24 \ _{i} \ _{i1} (1 \ ) + 4 \ _{i} \ _{i1} ;$$

which is positive. And at $= 1$; it becomes

$$12 \ _{i1} (1 + \ ) + \ _{i1} ^{2} + 12 \ _{i} (1 + 2 \ ) + \ _{i} ^{2} (13 + \ ) \ h$$

$$4 \ _{i} \ _{i1} (6 + \ ^{2} + 6);$$

which is negative if

$$(10) \qquad h \quad \frac{4 \ _{i} \ _{i1} (6 + \ ^{2} + 6)}{(12 \ _{i1} (1 + \ ) + \ _{i1} ^{2} + 12 \ _{i} (1 + 2 \ ) + \ _{i} ^{2} (13 + \ ))}:$$

Thus, one of the root of lies in [1 ; 1+] under the condition on :

Now we show that the matrix is diagonally dominant with the diagonal element . We have

$$= (\ _{i1} + 10 \ _{i} \ _{i}) h + 4 \ _{i} \ _{i1} (\ 3)$$

and this is negative if

$$h \quad \frac{4 \ _{i} \ _{i1} (3 \ )}{(\ _{i1} + 10 \ _{i} \ _{i})}:$$

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This is true for $\beta \in [-1-\delta, -1+\delta]$ if

(11) $$h \leq \frac{12\alpha_i\alpha_{i-1} - 4\alpha_i\alpha_{i-1}(-1+\delta)}{\alpha_{i-1} + 10\alpha_i - \alpha_i(-1-\delta)}.$$

Next we see that from the hypothesis (2) p is also negative.

Obviously s is positive.

We have

$$(|\lambda| - |p| - |\frac{s}{\beta}| \geq (|\lambda| - |p| - \frac{s}{-1+\delta})$$

therefore

(12) $$D = -\frac{-2\alpha_{i-1} + \alpha_{i-1}\delta - 10\alpha_i + 11\alpha_i\delta - \alpha_i\delta^2}{-1+\delta}h$$
$$-\frac{8\alpha_i\alpha_{i-1} - 16\alpha_i\alpha_{i-1}\delta + 4\alpha_i\alpha_{i-1}\delta^2}{-1+\delta}.$$

This is positive if

(13) $$h < \frac{4\alpha_i\alpha_{i-1}(2 - 4\delta + \delta^2)}{\alpha_i(10 + \delta^2 - 11\delta) + (2-\delta)\alpha_{i-1}}.$$

We see that the expression on the right hand side of the inequality (10) is smaller than the expressions on the right hand side of the inequalities (11) and (13). Hence the result is true if

$$h \leq \frac{4\alpha_i\alpha_{i-1}\delta(6\delta + \delta^2 + 6)}{(12\alpha_{i-1}(1+\delta) + \alpha_{i-1}\delta^2 + 12\alpha_i(1+2\delta) + \alpha_i\delta^2(13+\delta))}.$$

To visualize the above condition we take, for example, $\delta = 0.1$ in this case the above condition becomes

$$h \leq \frac{2.644\alpha_i\alpha_{i-1}}{13.21\alpha_{i-1} + 14.531\alpha_i}.$$

ERROR OF APPROXIMATION :

We write $e(x) = s(x) - f(x)$. Since $s^{'}(x_i) = M_i$, i = 1,2,...,n, we get $e_i^{'} = M_i - f_i^{'}$.

From (9) we find

(14) $$pe'_{i-2} + qe'_{i-1} + re'_i + se'_{i+1} = W_i - pf'_{i-2} - qf'_{i-1} - rf'_i - sf'_{i+1}, \text{ i = 1,2,...,n.}$$

We denote the right hand side of the above expression by U_i, say. We have

$$U_i = 12h^{-2}[\alpha_{i-1}(h + 2\alpha_i)(F_{i+1} - F_i) + \alpha_i(h - 2\alpha_{i-1})(F_i - F_{i-1})] - (h\alpha_i - 4\alpha_i\alpha_{i-1}) f'_{i-2} - (h\alpha_{i-1} - 12\alpha_i\alpha_{i-1} + 10h\alpha_i) f'_{i-1} - (10h\alpha_{i-1} + 12\alpha_i\alpha_{i-1} + h\alpha_i) f'_i - (h\alpha_{i-1} + 4\alpha_i\alpha_{i-1}) f'_{i+1}.$$

By Taylor's series expansion, we see that

$$F_{i+1} - F_i = \int_{x_i}^{x_{i+1}} (x - x_i) \left(f'(\xi_{i+1}(x)) - f'_{i+1}\right) dx + \int_{x_{i-1}}^{x_i} (x_i - x) \left(f'(\eta_i(x)) - f'_i\right) dx + \frac{h^2}{2} f'_{i+1} + \frac{h^2}{2} f'_i.$$

$$U_i = h(12h^{-2}((\int_{x_i}^{x_{i+1}} (x - x_i) \left(f'(\xi_{i+1}(x)) - f'_{i+1}\right) dx + \int_{x_{i-1}}^{x_i} (x_i - x) \left(f'(\eta_i(x)) - f'_i\right) dx)\alpha_{i-1} + (\int_{x_{i-1}}^{x_i} (x - x_{i-1}) \left(f'(\xi_i(x)) - f'_i\right) dx + \int_{x_{i-2}}^{x_{i-1}} (x_{i-1} - x) \left(f'(\eta_{i-1}(x)) - f'_{i-1}\right) dx)\alpha_i) + 5\alpha_{i-1}f'_{i+1} - 4\alpha_{i-1}f'_i + 5\alpha_i f'_i - 4\alpha_i f'_{i-1} - \alpha_i f'_{i-2} - \alpha_{i-1}f'_{i-1}) + 24h^{-2}\alpha_i\alpha_{i-1}(\int_{x_i}^{x_{i+1}} (x - x_i) \left(f'(\xi_{i+1}(x)) - f'_{i+1}\right) dx) + \int_{x_{i-1}}^{x_i} (x_i - x) \left(f'(\eta_i(x)) - f'_i\right) dx - \int_{x_{i-1}}^{x_i} (x - x_{i-1}) \left(f'(\xi_i(x)) - f'_i\right) dx - \int_{x_{i-2}}^{x_{i-1}} (x_{i-1} - x) \left(f'(\eta_{i-1}(x)) - f'_{i-1}\right) dx + 12\alpha_i\alpha_{i-1} \left(f'_{i+1} - f'_i\right) + 4\alpha_i\alpha_{i-1} \left(f'_{i-2} - f'_{i+1}\right).$$

Hence

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$$|U_i| \leq h \left(\begin{array}{c} 12\alpha_{i-1}\omega\left(f^{'};h\right) + 12\alpha_i\omega\left(f^{'};h\right) + 4\alpha_{i-1}\left(f^{'}_{i+1} - f^{'}_i\right) \\ +4\alpha_i\left(f^{'}_i - f^{'}_{i-1}\right) + \alpha_{i-1}\left(f^{'}_{i+1} - f^{'}_{i-1}\right) + \alpha_i\left(f^{'}_i - f^{'}_{i-2}\right) \end{array} \right)$$

$$+ 72\alpha_i\alpha_{i-1}\omega\left(f^{'};h\right),$$

$$\leq 18((\alpha_{i-1} + \alpha_i)\,h + 4\alpha_i\alpha_{i-1})\omega\left(f^{'};h\right).$$

If we take $\alpha = \max\{\alpha_i\}$

then

$$|U_i| \leq 36\alpha\,(h + 2\alpha)\,\omega\left(f^{'};h\right).$$

Hence

$$||e^{'}_i|| = \sup|e^{'}_i| \leq ||A^{-1}||||U_i||$$

We have

$$||e^{'}_i|| \leq 18((\alpha_{i-1} + \alpha_i)\,h + 4\alpha_i\alpha_{i-1})D^{-1}\omega\left(f^{'};h\right)$$

where D is defined in (12).

From the condition (1) there is at least one point $x^{'}_{i-1}$ in $[x_{i-1}, x_i]$ such that $e(x^{'}_{i-1}) = 0$. This gives that

$$e(x) = \int_{x^{'}_{i-1}}^{x} e^{'}(x)dx.$$

Therefore

$$|e(x)| \leq h\left(18((\alpha_{i-1} + \alpha_i)\,h + 4\alpha_i\alpha_{i-1})D^{-1}\right)\omega\left(f^{'};h\right).$$

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