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Classification of PS – Algebras

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Abstract

In this paper, we define a new algebra, namely PS – Algebra along with PS- ideal and discussed some of their properties in detail.

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Keywords : PS-algebra, PS-sub algebra, PS-ideal, G-part, p-semi simple and medial of PS-algebra.

1. Introduction

K.Iseki and S.Tanaka **[4]** introduced the concept of BCK-algebras in 1978. K.Iseki **[5]** introduced the concept of BCI-algebras in 1980. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J.Neggers, S.S.Ahn and H.S.Kim **[2]** introduced a notion called d-algebras, which is a generalization of BCK/BCI/BCH-algebras, and generalized some theorems from the theory of BCI-algebras.K.Megalai and A.Tamilarasi **[6]** introduced a new notion ,TM-algebras, which is a generalization of BCK/BCI/BCH/BCC/Q-algebras.C.Prabayak**[1]** and U.Leerawat introduced the concept of KU-algebra. In this paper, the new notion, PS-algebra, PS- sub algebra, PS-ideals are introduced. More over we established the properties of PS-algebras through R-closed ideals, homomorphism, G-part, p-radical and medial .

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1 [4,9]

A BCK- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $(x * y) * (x * z) \leq (z * y)$

ii) $x * (x * y) \leq y$

iii) $x \leq x$

iv) $x \leq y$ and $y \leq x \Rightarrow x=y$

v) $0 \leq x \Rightarrow x=0$, where $x \leq y$ is defined by $x * y = 0$,for all $x, y, z \in X$.

Definition 2.2 [5]

A BCI- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $(x * y) *(x * z) \leq (z*y)$

ii) $x * (x * y) \leq y$

iii) $x \leq x$

iv) $x \leq y$ and $y \leq x \Rightarrow x = y$

v) $x \leq 0 \Rightarrow x = 0$, where $x \leq y$ is defined by $x * y = 0$,for all $x, y, z \in X$.

Definition 2.3 [3]

A Q- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

(i) $x * x = 0$

(ii) $x * 0 = x$

(iii) $(x * y)*z = (x * z) * y$, where $x \leq y$ is defined by $x * y = 0$, for all $x, y, z \in X$.

Definition 2.4 [2]

A d- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $x * x = 0$

ii) 0 * x = 0

iii) x * y = 0 and y * x = 0 imply x = y , for all x, y ∈ X.

Definition 2.5 [1]

A KU- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) (x * y) * ((y * z) * (x * z)) =0

ii) x * 0 = 0

iii) 0 * x = x

iv) x * y = 0 and y * x = 0 imply x = y for all x, y, z ∈ X.

Remark :

- Every BCK-algebra is a TM-algebra but not the converse.
- Every BCK-algebra is a BCI-algebra but not the converse.
- Every BCI-algebra is a BCH-algebra but not the converse.
- Every BCH-algebra is a Q-algebra but not the converse.
- Every TM-algebra is a BH-algebra but not the converse.
- Every BCK-algebra is a d-algebra but not the converse.

3. PS-Algebras and its Properties

In this section we define PS-algebra, PS-ideal and discussed some of its properties.

Definition 3.1

A nonempty set X with a constant 0 and a binary operation ' ∗ ' is called PS – algebra if it satisfies the following axioms.

1. x * x = 0
2. x * 0 = 0
3. x * y = 0 and y * x = 0 ⇒ x = y , ∀ x ,y ∈ X.

Definition 3.2 :

Let S be a non empty subset of a PS -algebra X , then S is called a PS-sub algebra of X if

x ∗ y ∈ S ,for all x ,y ∈ S.

Definition 3.3 :

Let X be a PS-algebra and I be a subset of X, then I is called a PS-ideal of X if it satisfies following conditions:

1. 0 ∈ I
2. y ∗ x ∈ I and y ∈ I ⇒ x ∈ I

Example 3.4 : Let X = { 0, 1, 2 } be the set with the following table.

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

Then (X , ∗ , 0) is a PS – Algebra.

Example 3.5 : Let X = { 0,a,b } be the set with the following table.

*	0	a	b
0	0	a	b
a	0	0	0
b	0	b	0

Then (X , ∗ , 0) is a PS – Algebra.

Proposition 3.6 : In any PS-algebra (X , *,0), with x ≤ y, the following holds good for all x , y ∈ X.

1. x * (y * x) = y * (x * x)
2. y * (x * (y * x)) = 0
3. y * (y * (y * x)) = y * x
4. y * (x * (x * y)) = 0.
5. (x * y) * 0 = (x * 0) * (y * 0)
6. (x * y) * x = (x * x) * y

Proof :

(i) y*(x*x) = y*0 by 1 of definition 3.1

= 0 by 2 of definition 3.1

= x*0 by 2 of definition 3.1

= x*(y*x)

(ii) y*(x*(y*x)) = y*0 by (i) of 3.6

= 0 by 2 of definition 3.1

(iii) y*(y*(y*x)) = y*(y*0)

= y*0 by 2 of definition 3.1

= 0 by 2 of definition 3.1

= y*x

(iv) y*(x*(x*y)) = y*(x*0)

= y*0 by 2 of definition 3.1

= 0 by 2 of definition 3.1

(v) (x * y) * 0 = 0 by 2 of definition 3.1

= 0 * 0 by 1 of definition 3.1

= (x * 0) * (y * 0) by 2 of definition 3.1

(vi) (x * y) * x = 0 * x (by 3 of definition 3.1)

= 0 * y (by 3 of definition 3.1)

= (x * x) * y (by 1 of definition 3.1)

Remark :

1. A BCI algebra is a PS-algebra (X,*,0) satisfying the following additional axioms :

   (1) (x*y)*(x*z)*(z*y) = 0
   (2) (x*(x*y)*y) = 0

2. A KU- algebra is a PS-algebra (X,*,0) satisfying the following additional axioms :

   (1) (x*y)*((y*z)*(x*z)) = 0
   (2) 0*x = x

Proposition 3.7 : Let X be a PS-algebra. If x ≠ y and y * x = 0 , then x * y ≠ 0.
Proof : It is straightforward.

Example 3.8: Let X = { 0, a, b } be a PS-algebra, Which is not a BCK / d - algebra

*	0	a	b
0	0	a	b
a	0	0	a
b	0	b	0

Since (i) 0*a ≠ 0 (ii) a*(a*b)*b =b ≠ 0.

Example 3.9 : Let X = { 0,a,b,c } be a PS-algebra , which is not a KU / TM -algebra.

*	0	a	b	c
0	0	b	a	c
a	0	0	0	b
b	0	0	0	b
c	0	b	b	0

Since i) 0 * a = b ≠ a, and ii) (a*0)*((0*c)*(a*c)) = a ≠ 0.

Example 3.10 : Let X = { 0,a,b,c } be a PS-algebra, which is not a BRK – algebra.

*	0	a	b	c	d
0	0	b	a	c	d
a	0	0	0	b	c
b	0	0	0	b	a
c	0	b	b	0	b
d	0	c	b	a	0

Since (i) a * 0 = 0≠ a (ii) (a*b)*a ≠ 0 * b

Example 3.11: Let X = { 0,a,b,c } be the set with the following Cayley table.

*	0	a	b	c
0	0	b	a	c
a	0	0	0	b
b	0	0	0	b
c	0	b	b	0

It is clear that (X,*,0) is a PS – Algebra, but not a KU-algebra, Since i) 0 * a = b ≠ a, and

ii) (a*0)*((0*c)*(a*c)) = a ≠ 0.

It is easy to see that BCK / TM/d/ KU / Q /BRK – algebras**[8]** and PS-algebras are different notions.

Remark : From example 3.8 , I = {0,a} is a PS-ideal.
From example 3.10 , I_1 = {0,a,b} and
I_2 = {0,d} are PS-ideals.

Example 3.12 : Let X = { 0,a,b,c } be a PS-algebra .

*	0	a	b	c
0	0	b	0	c
a	0	0	0	b
b	0	0	0	b
c	0	b	b	0

Here I = { 0,b,c} is a PS-ideal.

Proposition 3.13 :

Let I be a PS-ideal of a PS-algebra X. If $x \in I$ and $x * y = 0$, then $y \in I$.

Proof: Let $x \in I$ and $x * y = 0$.

$\Rightarrow x \in I$ and $x * y \in I$. Since I is a PS-ideal we have $y \in I$.

Proposition 3.14 :

Let X be a PS-algebra and I be a non-empty subset of X containing 0. Then I is a PS-ideal of X iff

$y * x \in I, x \notin I \Rightarrow y \notin I$, for all $x, y \in X$.

Proof: Let I be a PS-ideal of X and $y * x \in I$ and $x \notin I$. Suppose that $y \in I$, then $x \in I$, since I is a PS-ideal, which is a contradiction.$\Rightarrow y \notin I$.

Conversely, Let us assume that $y * x \in I, x \notin I \Rightarrow y \notin I$, for all $x, y \in X$.

If $y * x \in I$ and $y \in I$, then it is clear that $x \in I$. Hence I is a PS-ideal.

Corollary 3.15 :

Let X be a PS-algebra and I be a non-empty subset of X containing 0.Then

(i) I is a PS-ideal of X iff $y * x \in I, y \notin I \Rightarrow x \notin I$, for all $x, y \in X$.

(ii) I is a PS-ideal of X iff $y * x \in I, x \in I \Rightarrow y \in I$, for all $x, y \in X$.

(iii) I is a PS-ideal of X iff $x \in I, y \in I \Rightarrow y * x \in I$, for all $x, y \in X$.

Remark:

Let X be a PS-algebra and I be a non-empty subset of X containing 0. Then

(i) I is a PS-ideal of X iff I is a ideal of X.

(ii) I is a PS-ideal of X iff I is a BCK- ideal of X.

(iii) I is a PS-ideal of X iff I is a PS- Sub algebra of X.

Definition 3.16 : An ideal I of a PS-algebra X is said to be R-closed if $x * 0 \in I$ for all $x \in I$.

Theorem 3.17: Every R-closed ideal of a PS-algebra is a PS-sub algebra.

Proof : Let I be a R-closed ideal of a PS- algebra (X,*,0).

Let $x, y \in I$. Then $x * 0$ and $y* 0 \in I$.

In PS-algebra , $(x * y) * 0 = 0$. Hence $x*y \in I$.

So I is a sub-algebra of PS-algebra.

Definition 3.18

A mapping $f: X \rightarrow Y$ of a PS-algebra is called a homomorphism if $f (x * y) = f (x) \Delta f (y)$,$\forall x, y \in X$.

Remark :

If $f: X \rightarrow Y$ is a homomorphism of PS-algebra, then $f (0) = 0'$.

Theorem 3.19: If $f: X \rightarrow Y$ is a homomorphism of PS-algebra, then (i) $f (0) = 0'$ and (ii) If $x * y = 0$, $\forall x, y \in X$ then $f(x) \Delta f(y) = 0'$.

Proof: (i) $f (0) = f (0 * 0) = f (0) = f (0) \Delta f (0) = 0'$

(ii) Let $x, y \in X$ and $x * y = 0$.

$$\begin{aligned} \text{Then } f (x) \Delta f (y) &= f (x * y) \\ &= f (0) \\ &= 0'. \end{aligned}$$

Theorem 3.20 : Let (X,*, 0), (Y, Δ, 0') be PS-algebras and let J be a PS- ideal of Y. Let $f: X \rightarrow Y$ be a homomorphism. Then $f^{-1}(J)$ is a PS- ideal of X.

Proof:

We know that $f^{-1} (J) =\{x \in X / f (x) = y \text{ for } y \in J \}$.

Since $0' \in J$ and $f (0) = 0'$, $0 \in f^{-1}(J)$.

Assume $y * x \in f^{-1} (J)$ and $y \in f^{-1} (J)$, then $f (y * x) \in J$ and $f (y) \in J$.

Since f is a homomorphism $f (y * x) = f (y) \Delta f (x) \in J$.

Since J is a PS- ideal of Y, $f (x) \in J \Rightarrow x \in f^{-1} (J)$.

Hence $f^{-1} (J)$ is a PS- ideal of X.

Definition 3.21 : The set $\{x \in X / f(x) = 0'\}$ is known as kernel of f and is denoted by ker f.

Theorem 3.22: Let f: X→ Y is an anti homomorphism of PS- algebra then ker f is a PS- ideal of X.

Proof: Obviously $0 \in \ker f \Rightarrow f(0) = 0'$.

Let $y * x \in \ker f$ and $y \in \ker f$.

$\Rightarrow f(y * x) = 0'$ and $f(y) = 0'$

$\Rightarrow f(x) \Delta f(y) = 0'$

$\Rightarrow f(x) \Delta 0' = 0'$

$\Rightarrow f(x) = 0'$

$\Rightarrow x \in \ker f$.

Proposition 3.23: Let $f : (X,*, 0) \rightarrow (Y,\Delta,0')$ be an homomorphism of PS-algebra, then

(i) If S is a PS - sub algebra of X, then f(S) is a PS – sub algebra of Y.

(ii) If S is a PS - sub algebra of Y, then $f^{-1}(S)$ is a PS – sub algebra of X.

4. G-part , p-radical and medial of PS-algebras

Definition 4.1 : Let X be a PS-algebra. For any subset S of X, we define $G(S) = \{x \in S \,/\, 0 * x = x\}$

In particular, if S = X, then we say that G(X) is the G-part of a PS-algebra.

Definition 4.2: Let X be a PS-algebra.Then the set $B(X)=\{x\in X \,/\, 0 * x = 0\}$is called a p-radical of X.

Proposition 4.3 : Let X be a PS-algebra. The p- radical B(X) of X is PS- sub algebra.

Proof: By definition of PS- sub algebra, it is obvious that B(X) is PS- sub algebra of X.

Theorem 4.4 : Let (X,*,0) be a PS-algebra. The p-radical B(X) of X is a PS-ideal of X.

Proof: Since $0 * (0 * 0) = 0$, $0 \in B(X)$.

Let $y * x \in B(X)$ and $y \in B(X)$.

Then it implies $0 * (y * x) = 0$ and $0 * y = 0$.

Now, $(0 * y) * (0 * x) = 0$

$\Rightarrow 0 * (0 * x) = 0$

$\Rightarrow (0 * 0) * x = 0$

$\Rightarrow 0 * x = 0$

$\Rightarrow x \in B(X)$

$\therefore$ B(X) is an ideal of X.

Definition 4.5 : A PS-algebra X is said to be p-semi simple if $B(X) = \{0\}$.

Remark : It is obvious that $G(X) \cap B(X) = \{0\}$.

From example 3.2, Let us consider $G(X) = \{0,1,2\}$ and $B\{X\}=\{0\}$.

It is clear that $G(X) \cap B(X) = \{0\}$.

Theorem 4.6: Let (X,*, 0) be a PS-algebra. If G(X) = X, then X is p-semi simple.

Proof: Let G(X) = X.

Also $G(X) \cap B(X) = \{0\}$

$\Rightarrow X \cap B(X) = \{0\}$, (i.e) $B(X) = \{0\}$.

Hence X is P-semi simple.

Theorem 4.7: If S is a PS- sub algebra of a PS-algebra (X,*, 0), then $G(X) \cap S = G(S)$.

Proof: Obviously , $G(X) \cap S \subset G(S)$.

We know that $G(S) = \{x \in S \subset X \,/\, 0 * x = x\}$.

Let $x \in G(S)$. Then $0 * x = x$ and $x \in S \subset X$ which implies that $x \in G(X) \cap S$.

Hence $G(S) \subset G(X) \cap S$.

Therefore $G(X) \subset S = G(S)$.

Theorem 4.8: If (X,*,0) is a PS – algebra and $x, y \in X$ then $y \in G(X) \Leftrightarrow (x * y) * x = y$.

Proof: Let (X,*,0) be a PS-algebra. Let $x, y \in X$ and $y \in G(X)$. Then $0 * y = y$.

Now, $(x * y) * x = (x * x) * y = 0 * y = y$

$\therefore (x * y) * x = y$.

Conversely, Let us assume that $(x * y) * x = y$.

Now, $0 * y = (x * x) * y = (x * y) * x = y$

$\Rightarrow y \in G(X)$.

Theorem 4.9: If (X,*,0) is a PS – algebra and $x, y \in X$ then $y \in B(X) \Leftrightarrow (x * y) * x = 0$.

Proof: Let (X,*,0) be a PS-algebra. Let x, y $\in$X and y $\in$B(X). Then 0 * y = 0.

Now, (x * y) * x = (x * x) * y = 0 * y = 0.

$\therefore$ (x * y) * x = 0.

Conversely, Let us assume that (x * y) * x = 0.

Now, 0 * y = (x * x) * y = (x * y) * x = 0

$\Rightarrow$ y $\in$B(X).

Theorem 4.10: If (X,*,0) is a PS – algebra and x * y = x * z then 0 * y = 0 * z, where x, y, z$\in$X.

Proof : Now, (x * y) * x = (x * x) * y = 0 * y

(x * z) * x = (x * x) * z = 0 * z

Since x * y = x * z, then 0 * y = 0 * z.

Theorem 4.11: Let X be a PS algebra. Then the left cancellation law holds in G(X).

Proof: Let x,y,z $\in$ G(X) with x * y = x * z.

By the above theorem, 0 * y = 0 * z.

Since y, z $\in$ G(X),we obtain y = z.

Theorem 4.12 : Let (X,*,0) be a PS-algebra. Then x $\in$ G(X) iff 0 * x $\in$ G(X).

Proof: If x $\in$ G(X) , then 0 * x = x .

Now, 0 * (0 * x) = 0 * x .

$\Rightarrow$ 0 * x $\in$G(X).

Conversely, If 0 * x $\in$ G(X), then 0 * (0 * x) = 0 * x.

By left cancellation, we get 0 * x = x.

$\Rightarrow$ x $\in$ G(X).

Corollary 4.13: If (X,*,0) is a PS-algebra of order 3,then |G(X)| $\neq$ 3,that is G(X) $\neq$ X.

Theorem 4.14: Let (X,*,0) be a PS-algebra of order 3.Then G(X) is a PS-ideal of X if and only if |G(X)| = 1.

Proof : Let X= {0,a,b} be a PS-algebra. If |G(X)| = 1,then G(X) ={0} is the trivial ideal of X.

Conversely, assume that G(X) is an ideal of X. By corollary 4.13,we know that either |G(X)| = 1 or

|G(X)| = 2. Suppose that |G(X)| = 2. Then either

G(X) = {0,a} or G(X) = {0,b}.

If G(X) = {0,a}, then b*a $\notin$G(X) because G(X) is an ideal of X so b*a = b

Now 0 = a*0 = a*(b*b) = b*(a*b) = b*a = b, a contradiction. Similarly G(X) = {0,b} leads to a contradiction. Therefore |G(X)| $\neq$ 2 and so |G(X)| = 1.

Definition 4.15 : Let (X,*,0) be a PS-algebra satisfying (x*y)*(z*u) = (x*z)*(y*u) for any x,y,z,u $\in$X is called a medial of PS-algebra.

Theorem 4.16 : Let (X,*,0) be a medial of PS-algebra. Then G(X) is a PS-sub algebra of X.

Proof : Let x, y $\in$G(X) . Then 0 * x = x and 0*y = y.

Hence 0*(x*y) = (0*0) * (x*y)

= (0*x) * (0*y)

= x*y.

Theorem 4.17: Let (X,*,0) be a medial of PS-algebra. Then B(X) is a PS-sub algebra of X.

Proof : Let x, y $\in$B(X) . Then 0 * x = 0 and 0*y = 0.

Hence 0*(x*y) = (0*0) * (x*y)

= (0*x) * (0*y)

= 0.

Theorem 4.18: Let (X,*,0) be a medial of PS-algebra. Then (i) (x*y)*z = y*z (ii) x*(y*z) = y*(x*z) (iii) x*y = 0,for any x,y,z $\in$G(X) .

Proof : (i) (x*y)*z = (x*y)*(0*z)

= (x*0)*(y*z)

= 0*(y*z)

= y*z

(ii) x*(y*z) = (0*x)*(y*z)

= (0*y)*(x*z)

= y*(x*z)

(iii) x*y = 0*(x*y)

= (y*y)*(x*y)

= (y*x)*(y*y)

= (y*x)*0

= 0

Theorem 4.19: Let (X,*,0) be a medial of PS-algebra. Then G(X) is a PS-ideal of X.

Proof : Since 0*0 = 0 by(i) of 3.1 $\Rightarrow$ 0 $\in$G(X).

Let x,y $\in$X be such that y * x $\in$ G(X) and y $\in$ G(X).

Then it implies 0 * (y * x) = y*x and 0 * y = y.

Now, y = (x*y)*x , by theorem 4.8

Put y = x , we get

x = (y*x)*y = (0*(y*x))*(0*y)= (0*0)*((y*x)*y) = 0 *x .

Thus we get 0*x = x. Hence x ∈G(X).

Therefore G(X) is a PS-ideal of X.

Definition 4.20: An ideal I of a PS-algebra (X,*,0) is said to be closed if 0*x ∈I for all x ∈I.

Theorem 4.21: Let (X,*,0) be a medial of PS-algebra. Every closed ideal of X is a PS-sub algebra.

Proof : Let (X,*,0) be a medial of PS-algebra. Let I be a closed ideal of X.
Let x, y ∈I. Then 0 * x and 0 * y ∈ I.
Now, 0 * (x*y) = (0*0) * (x * y) = (0*x) * (0*y) and hence 0*(x*y) ∈I. Hence x*y ∈I.
So I is a sub-algebra of PS-algebra.

5. Conclusion

The concept of proposed PS-algebra in this work, has been evaluated against well established algebraic theorems. It has been observed that PS-algebra; satisfies the various conditions stated in BCK/BCI/BCH /Q/TM/KU algebras and hence it can be considered as the generalization of all these algebras.

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