IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 1 Issue 7, September 2014.
www.ijiset.com
ISSN 2348 – 7968

Integral points on the hyperbola $x^2 - 4xy + y^2 + 15x = 0$

S.Vidhyalakshmi[1], A.Kavitha[2], M.A.Gopalan[3]
[1,2,3]Professor, Department of Mathematics, Shrimati Indira Gandhi College,Trichy.

ABSTRACT

This paper concerns with the problem of obtaining infinitely many non-zero distinct integer solutions of the binary quadratic Diophantine equation representing hyperbola given by $x^2 - 4xy + y^2 + 15x = 0$. Employing the lemma of Brahmagupta, infinitely many integral solutions of the above equation are obtained. The recurrence relations on the solutions are presented. A few interesting relations among the solutions are also given.

Key words: Binary quadratic, Hyperbola, Pell equation, Integer solutions

2010 Mathematics subject classification: 11D09

INTRODUCTION

The binary quadratic Diophantine equations offer an unlimited field for research because of their variety [1,2]. For an extensive review of various problems one may refer [3-21].This communication concerns with yet another interesting binary quadratic equation $x^2 - 4xy + y^2 + 15x = 0$ representing hyperbola for determining its infinitely many non zero integral solutions. Also, a few interesting relations among the solutions are presented.

Method of analysis

The Diophantine equation to be solved for its non-zero distinct integral solution is

$$x^2 - 4xy + y^2 + 15x = 0 \qquad (1)$$

Treating (1) as a quadratic in y, we get

$$y = 2x \pm \sqrt{3x^2 - 15x} \qquad (2)$$

Let $\alpha^2 = 3x^2 - 15x \qquad (3)$

Using (3) in (2) we have

$$X^2 = 12\alpha^2 + 15^2$$

where $X = 6x - 15 \qquad (4)$

The initial solution of (4) is

$\alpha_0 = 30 \quad X_0 = 105$

Now consider the Pell equation

$$X^2 = 12\alpha^2 + 1 \qquad (5)$$

whose fundamental solution is $(\widetilde{\alpha_0}, \widetilde{X_0})$=(2,7). The other solutions of (5) can be derived from the relations

$$\widetilde{X_n} = \frac{f_n}{2} \text{ and } \widetilde{\alpha_n} = \frac{g_n}{4\sqrt{3}}$$

Where

$$f_n = \left[\left(7 + 4\sqrt{3}\right)^{n+1} + \left(7 - 4\sqrt{3}\right)^{n+1}\right]$$

$$g_n = \left[\left(7 + 4\sqrt{3}\right)^{n+1} - \left(7 - 4\sqrt{3}\right)^{n+1}\right],$$

n=0,1,2,3……..

Applying the lemma of Brahmagupta between (α_0, X_0) and $(\widetilde{\alpha_n}, \widetilde{X_n})$, the other solutions of (4) can be obtained from the relations

$$\left.\begin{aligned} \alpha_{n+1} &= 15f_n + \frac{105}{4\sqrt{3}} g_n \\ X_{n+1} &= 105\frac{f_n}{2} + \frac{90}{\sqrt{3}} g_n \end{aligned}\right\} \qquad (6)$$

Taking the positive sign in the RHS of (2) and using (4) and (6), the non-zero distinct integer solutions of the hyperbola (1) are represented by

$$\left.\begin{aligned} x_{n+1} &= \frac{1}{6}\left[\frac{105f_n}{2} + \frac{90g_n}{\sqrt{3}} + 15\right] \\ y_{n+1} &= \frac{65f_n}{2} + \frac{225g_n}{4\sqrt{3}} + 5 \end{aligned}\right\} \quad (7)$$

where n= 0,1,2,3…….

Some numerical examples are presented below:

n	x_{n+1}	y_{n+1}
0	245	910
1	3380	12610
2	47045	175570
3	655220	2445310

The recurrence relations satisfied by the solutions of (1) are given by

$$y_{n+3} - 14y_{n+2} + y_{n+1} = -60$$

$$x_{n+3} - 14x_{n+2} + x_{n+1} = -30$$

A few interesting relations among the solutions are as follows:

1. $57699216x_{n+3} - 26217138240x_{n+1} - 1382599980y_{n+1} = 4094365560$

2. $57699216x_{n+2} - 131261918x_{n+1} + 2431915104y_{n+1} = -2762817120$

3. $57699216y_{n+2} - 9400057596x_{n+1} - 4145325y_{n+1} = 4932633489$

4. $57699216y_{n+3} - 124815485244x_{n+1} - 10176714y_{n+1} = 49020508750$

Also, taking the negative sign in the R.H.S of (2)

$$x_{n+1} = \frac{1}{12}[105f_n + 60\sqrt{3}g_n + 30]$$

$$y_{n+1} = \frac{1}{12}[30f_n + 15\sqrt{3}g_n + 60]$$

n=0,1,2,3………..

PROPERTIES:

1. $60x_{n+2} - 900x_{n+1} - 240y_{n+1} = -3000$

2. $60x_{n+3} - 12540x_{n+1} - 3360y_{n+1} = -43800$

3. $60y_{n+2} - 240x_{n+1} - 60y_{n+1} = -98875$

4. $60y_{n+3} - 3360x_{n+1} - 900y_{n+1} = -11475$

CONCLUSION

As the binary quadratic Diophantine equations are rich in variety, one may consider other choices of hyperbolas and search for their patterns of solutions and their corresponding properties.

ACKNOWLEDGEMENT

The finical support from the UGC, New Delhi F.MRP-5123/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged

REFERENCES

1. L.E.Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing Company, New York (1952).
2. L.J.M Ordell, Diophantine equations, Academic Press, New York(1969).
3. Gopalan M.A.,Vidhyalakshmi.s and Devibala.S, “On the Diophantine equation $3x^2 + xy =$ 14”, Acta Ciencia Indica, Vol.XXXIIIM, No.2, Pg.645-646,2007
4. Gopalan.M.A, Janaki.G, “ Observations on $y^2 = 3x^2 + 1$ ”, Acta Ciencia Indica, Vol.IXXXIVM, No.2, Pg.693,2008
5. Gopalan.M.A, Vijalakshmi.R. “Special Pythagorean triangles generated through the integral solutions of the equation $y^2 = (K^2 + 1)x^2 + 1$ ”, Antarctica J.Math,7(5), Pg.503-507, 2010
6. Gopalan.M.A., and Sivagami.B, “Observations on the integral solutions of

$y^2 = 7x^2 + 1$ Antarctica J.Math,7(3), Pg.291-296, 2010

7. Gopalan.M.A, Vijalakshmi.R. "observation on the integral solutions of $y^2 = 5x^2 + 1$", Impact J.Sci.Tech, Vol.4,No.4,125-129,2010
8. Gopalan.M.A. and Sangeetha.G "A remarkable observation on $y^2 = 10x^2 + 1$" Impact J.Sci.Tech, Vol.4,No.1,103-106,2010
9. Gopala.M.A., Parvathy.G, "Integral points on the hyperbola $x^2 + 4xy + y^2 - 2x - 10y + 24 = 0$", Antarctica J.Math,7(2), Pg.149-155, 2010
10. Gopalan M.A. Palanikumar.R, "Observations on $y^2 = 12x^2 + 1$ Antarctica J.Math,8(2), Pg.149-152, 2011
11. Gopalan.M.A. Devibala.S, Vijayalakshmi.R, "Integral points on the Hyperbola $2x^2 - 3y^2 = 5$" American journal of Applied Mathematics and Mathematical Sciences, Vol.1, No.1,pg1-4,Jan-June,2012
12. Gopalan.M.A.,Vidhyalakshmi.S, Mallika.S,T.R.Usha Rani, "Observations On $y^2 = 12x^2 - 3$", Bessel J.Math2(3),Pg.153-158, 2012
13. Gopalan.M.A.,Vidhyalakshmi.S G.Sumathi,K.Lakshmi,"Integral points on the hyperbola $x^2 + 6xy + y^2 + 40x + 8y + 40 = 0$ ", Bessel J.Math2(3),Pg.159-164, 2012
14. Gopalan.M.A, Geetha.K, "Observation on the hyperbola$y^2 = 18x^2 + 1$", Retell, Vol.13,No.1,Pg.81-83, Nov.2012
15. Gopalan.M.A Sangeetha.G, Manju Somanath, "Integral points on the Hyperbola $(a + 2)x^2 - ay^2 = a(k - 1) + 2k^2$", Indian journal of science, Vol.1, No.2, pg.125-126,Dec.2012
16. Gopalan.M.A., Vidhyalakshmi.S, Kavitha.A,"Observations on the Hyperbola $ax^2 - (a + 1)y^2 = 3a - 1$ ", discovery, Vol.4, No.10,Pg.22-24, April-2013.
17. Meena.K, Vidhyalakshmi.S, N.Sujitha, Gopalan.M.A, "On the binary quadratic Diophantine equation $x^2 - 6xy + y^2 + 8x = 0$, Bulletin of Mathematics and Statistics Research" ,Vol.2,Issue.1,14-20,2014

18. Meena.K, Vidhyalakshmi.S, Gopalan.M.A, Nancy.T, "Integer points on the hyperbola $x^2 - 5xy + y^2 + 5x = 0$,Bulletin of Mathematics and Statistics Research" , Vol.2,Issue.1, 38-41,2014

19. Meena.K, Vidhyalakshmi.S, Gopalan.M.A, Nivetha.S, "Lattice points on the hyperbola $x^2 - 3xy + y^2 + 12x = 0$,Bessel.J.Math.,4(2)(2014),49-55

20. Meena.K, Vidhyalakshmi.S, Aarthy Thangam.S, Premalatha.E Gopalan.M.A, "Integer points on the hyperbola$x^2 - 6xy + y^2 + 4x = 0$,SJET" Vol.2,Issue.1, 14-18,2014

21. Meena.K, Vidhyalakshmi.S, Gopalan.M.A, Akila.G "Integer points on the hyperbola $x^2 - 10xy + y^2 + 8x = 0$, Bulletin of Mathematics and Statistics Research" , Vol.2,Issue.2,215-219,2014