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A New Class of Life time Distribution Based on Moment Generating Function Ordering with Hypothesis Testing

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Abstract: In this paper a new class of life distribution is derived based on moment generating function ordering, the class is called Exponential Better than Used in moment generating function ordering (EBU_{mgf}). A moment inequality for this class is derived and a test statistic for testing exponentiality against (EBU_{mgf}) is proposed based on this inequality. Critical values of this test are calculated. The power of the test and Pitman's asymptotic efficiency for some commonly used distributions in reliability are calculated. A set of real data is used as an example to elucidate the use of the proposed test statistic for practical reliability analysis.

Key Words: EBU_{mgf}, EWU_{mgf}, Exponential distribution, Moment Inequality, Pitman's Efficiency.

1. INTRODUCTION

Certain classes of life distributions and their variations have been introduced in reliability, the applications of these classes of life distribution can be seen in engineering, social, biological science and maintenance. Therefore, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions based on some aspects of aging. Concepts of aging describe how a population of units or systems improves or deteriorates with age. Many classes of life distributions are categorized and defined in literature according to some statistical ordering, see Yue and Cao (2001), Elbatal (2007), Ahmad and Sepehrifa (2009) and Kayid et al.(2010) .

Let X and Y be two nonnegative random variables, representing lives of an instrument with distribution functions $F(x)$ and $G(y)$, and their survival functions are $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, and their corresponding moment-generating functions are defined as

$$M_X(t) = \int_0^\infty e^{tx} dF(x), \quad M_Y(t) = \int_0^\infty e^{ty} dG(y) \text{ for all } t \geq 0$$

Definition 1.1: Klar and Muller (2003) showed that

Y is larger than X in moment-generating function order (denoted by$X \leq_{mgf} Y$) if $M_X(t) \leq M_Y(t)$ for all $t \geq 0$ so we can writ

$$\int_0^\infty e^{tx} \bar{F}(x) dx \leq \int_0^\infty e^{ty} \bar{G}(y) dy \text{ for all } t \geq 0$$

Definition 1.2: The non-negative random variable X with distribution F is said to be exponential better than used ordering (*EBU*) or we can write $X \in EBU$ (or $F \in EBU$), iff

$$\overline{F}_t(u) \leq e^{-u/\mu}$$

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Or equivalently

$$\bar{F}(u+t) \le \bar{F}(t)e^{-u/\mu} \quad \text{See Elbatal (2002).}$$

This paper is organized as follows, in section 2 the new class of life distribution based on the moment generating function ordering is introduced, a moment inequality is developed in section 3, a test statistic based on the inequality of the previous section for testing H_o: F is exponential against H_1: F is EBU_{mgf} (EWU_{mgf}) and not exponential is introduced in section 4. Pitman's asymptotic efficiency (PAE) of the test for some common distribution is tabulated in section 5. In section 6, Monte Carlo Critical points are obtained for sample sizes $n= 5(1)40$, 45 and 50 and power of the test is estimated in section 7. Finally, applications using real data are introduced in section 8.

2. The New Class EBU_{mgf} (EWU_{mgf}) of life distribution

In this section a new class of life distribution based on the moment-generating function ordering introduced in section 1 is presented, the new class called Exponential Better (Worth) than Used in moment-generating function order

Definition 2.1: A life distribution F and its survival function $\bar{F}$ is said to have the exponential better (worth) than used in moment-generating function order property EBU_{mgf} (EWU_{mgf}) if

$$\bar{F}(t)\int_0^\infty e^{\lambda x}e^{-x/\mu}\,dx \ge (\le) \int_0^\infty e^{\lambda x}\bar{F}(x+t)dx \quad \text{for all } x, t, \lambda \ge 0$$

Or equivalently

$$\frac{\mu}{1-\lambda\mu}\bar{F}(t) \ge (\le) \int_0^\infty e^{\lambda x}\bar{F}(x+t)dx$$

3. Moment Inequality

In the spirit of the work of Ahmad (2001), we state and prove the following result:

Theorem 3.1: If F is EBU_{mgf} (EWU_{mgf}), then for $r \ge 0$

$$\frac{\mu_{r+1}}{\lambda(r+1)(1-\lambda\mu)} \ge (\le) \frac{r!}{\lambda^{r+2}} E\left[e^{\lambda x} - \sum_{j=0}^{r}\frac{(\lambda x)^j}{j!}\right] \qquad (3.1)$$

Proof: F is EBU_{mgf} (EWU_{mgf}) if

$$\frac{\mu}{1-\lambda\mu}\bar{F}(t) \ge (\le) \int_0^\infty e^{\lambda x}\bar{F}(x+t)dx$$

Multiplying both sides by t^r and integrating w.r.t. t and x we get

$$\frac{\mu}{1-\lambda\mu}\int_0^\infty t^r\,\bar{F}(t)dt \geq (\leq)\int_0^\infty\int_0^\infty e^{\lambda x}t^r\bar{F}(x+t)dx\,dt$$

The *L.H.S* will be

$$L.H.S.=\frac{\mu}{1-\lambda\mu}\int_0^\infty t^r\,\bar{F}(t)dt$$

$$L.H.S.=\frac{\mu}{1-\lambda\mu}E\left[\int_0^X t^r\,dt\right]$$

$$=\frac{\mu}{1-\lambda\mu}E\left[\frac{t^{r+1}}{r+1}\right]_0^X$$

$$=\frac{\mu}{(1-\lambda\mu)(r+1)}\mu_{r+1} \qquad (3.2)$$

The *R.H.S* will be

$$R.H.S.=\int_0^\infty\int_0^\infty e^{\lambda x}t^r\bar{F}(x+t)dx\,dt$$

$$=E\int_0^X\int_0^{X-t}t^r e^{\lambda x}dxdt$$

$$=E\frac{1}{\lambda}\int_0^X t^r\left[e^{\lambda(x-t)}-1\right]dt$$

$$=\frac{1}{\lambda}E\left[\frac{e^{\lambda x}}{\lambda^{r+1}}\int_0^X(\lambda t)^r\,e^{-\lambda t}d\lambda t\right]-\frac{1}{\lambda(r+1)}E(X^{r+1})$$

$$=\frac{r!}{\lambda^{r+2}}E\left[e^{\lambda x}\int_0^X\frac{(\lambda t)^r}{r!}e^{-\lambda t}d\lambda t\right]-\frac{1}{\lambda(r+1)}\mu_{r+1} \qquad (3.3)$$

From (3.2) and (3.3) we cam writ that

$$\frac{\mu}{(1-\lambda\mu)(r+1)}\mu_{r+1}\geq(\leq)\frac{r!}{\lambda^{r+2}}E\left[e^{\lambda x}\int_0^X\frac{(\lambda t)^r}{r!}e^{-\lambda t}d\lambda t\right]-\frac{1}{\lambda(r+1)}\mu_{r+1}$$

Then

$$\frac{\mu}{(1-\lambda\mu)(r+1)}\mu_{r+1}\geq(\leq)\frac{r!}{\lambda^{r+2}}E\left[e^{\lambda x}-\sum_{j=0}^{r}\frac{(\lambda x)^j}{j!}\right]-\frac{1}{\lambda(r+1)}\mu_{r+1}$$

Or simply it can be written as

$$\frac{\mu_{r+1}}{\lambda(r+1)(1-\lambda\mu)}\geq(\leq)\frac{r!}{\lambda^{r+2}}E\left[e^{\lambda x}-\sum_{j=0}^{r}\frac{(\lambda x)^j}{j!}\right]$$

Then the proof is completed.

Corollary 3.1

Let $r = 0$ in (3.1) then

$$\frac{\mu}{\lambda(1-\lambda\mu)} \geq (\leq) \frac{1}{\lambda^2} E\left[e^{\lambda x} - 1\right] \qquad (3.4)$$

4. Testing Against EBU_{mgf} (EWU_{mgf}) Alternatives

The test presented in this section depends on a sample X_1, X_2, …..,X_n from a population with distribution F. the purpose is to test the null hypothesis H_o: F is exponential against H_1: F is EBU_{mgf} (EWU_{mgf}) and not exponential. Using the moment inequality obtained in theorem 3.1 and corollary 3.1, a measure of departure from H_o may be defined as follows:

$$\delta = \frac{\mu_{r+1}}{\lambda(r+1)(1-\lambda\mu)} - \frac{r!}{\lambda^{r+2}} E\left[e^{\lambda x} - \sum_{j=0}^{r} \frac{(\lambda x)^j}{j!}\right] \qquad (4.1)$$

The test can be written as H_o: $\delta = 0$ against H_1: $\delta > (<)\ 0$. The measure δ in (4.1) can be estimated by

$$\hat{\delta} = \frac{1}{n^2} \sum_{i=0}^{n} \sum_{k=0}^{n} \frac{X_k^{r+1}}{\lambda(1-\lambda\overline{X})(r+1)} - \frac{r!}{\lambda^{r+2}} [e^{\lambda x_i} - \sum_{j=0}^{r} \frac{(\lambda x_i)^j}{j!}] \qquad (4.2)$$

Let

$$\phi(X_1, X_2) = \frac{X_2^{r+1}}{\lambda(1-\lambda\mu)(r+1)} - \frac{r!}{\lambda^{r+2}} [e^{\lambda X_1} - \sum_{j=0}^{r} \frac{(\lambda X_1)^j}{j!}]$$

And define the symmetric Kernel

$$\psi(X_1, X_2) = \frac{1}{2!} \sum \phi(X_i, X_k)$$

Where th sum is over all the arrangement of X_i and X_k , then $\hat{\delta}$ is equivalent to U-Statistic given by

$$U_n = \frac{1}{\binom{n}{2}} \sum \phi(X_i, X_k)$$

Theorem 4.1:

As $n \to \infty$, $\sqrt{n}(\hat{\delta} - \delta)$ is asymptotically normal with mean 0 and variance σ^2, and under H_o the variance is σ_o^2 where

$$\sigma_0^{\ 2} = \frac{(r!)^2}{\lambda^{2r+4}} \left[\frac{1}{1-2\lambda} + \sum_{i=0}^{r} \sum_{j=0}^{r} \frac{\lambda^{i+j}}{i!\,j!}(i+j)! - 2\sum_{j=0}^{r} \frac{\lambda^j}{(1-\lambda)^{j+1}}\right] + \frac{(2r+2)!}{\lambda^2(1-\lambda)^2(r+1)^2}$$
$$-2\frac{r!}{\lambda^{r+3}(1-\lambda)(r+1)} \left[\frac{(r+1)!}{(1-\lambda)^{r+2}} - \sum_{j=0}^{r} \frac{\lambda^j}{j!}(r+j+1)!\right]$$

For $r = 0$ the variance reduces to

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$$\sigma_o^2\Big|_{r=0} = \frac{2}{(1-2\lambda)(1-\lambda)^3}, \qquad \lambda \neq \frac{1}{2}, 1$$

The proof follows from the standard theory of U-statistic Lee (1990) and direct calculations.

5- Pitman Asymptotic Efficiency (PAE)

The pitman asymptotic efficiency of the class EBU_{mgf} was calculated using the Linear Failure Rate (LFR), Makeham, and Weibull distributions. The Pitman efficiency is defined as:

$$PAE = \left(\frac{\partial \delta}{\partial \theta}\Big|_{\theta=\theta_o} \right) \Big/ \sigma_o$$

$$= \frac{1}{\sigma_o} \left| \frac{1}{\lambda(r++1)} \left[\frac{\mu'_{\theta(r+1)}}{1-\lambda\mu_\theta} + \mu_{\theta(r+1)} \frac{\lambda\mu'_\theta}{(1-\lambda\mu_\theta)^2} \right] + \frac{r!}{\lambda^{r+2}} \sum_{j=0}^{r} \frac{\lambda^j}{j!} \mu'_{\theta(j)} \right|$$

where μ' denote the partial derivative w.r.t. θ.

The following three families of alternatives are often used for efficiency calculation

- Linear Failure Rate (LFR) : $\overline{F}_\theta(x) = e^{-x-\frac{1}{2}\theta x^2}$
- Makeham : $\overline{F}_\theta(x) = e^{-x-\theta(x+e^{-x}-1)}$
- Weibull : $\overline{F}_\theta(x) = e^{-x^\theta}$

The null exponential is attained at $\theta = 0, 0$ and 1 respectively. The efficiency calculation for the above three alternatives at $r = 0$ are:

$$PAE(\delta)|_{LFR} = \frac{1}{\sigma_0} \left| \frac{-1}{\lambda(1-\lambda)^2} \right| \qquad (5.1)$$

$$PAE(\delta)|_{Mak} = \frac{1}{\sigma_0} \left| \frac{4\lambda^2-9\lambda+4}{2\lambda^2(1-\lambda)^2} \right| \qquad (5.2)$$

$$PAE(\delta)|_{Wieb} = \frac{1}{\sigma_0} \left| \frac{1.4228-\lambda^2}{\lambda^2(1-\lambda)^2} \right| \qquad (5.3)$$

The relations between efficiency and λ of the three distributions described in equations (5.1), (5.2) and (5.3) are plotted in Fig.(1) to determine the value of λ of the maximum efficiency.

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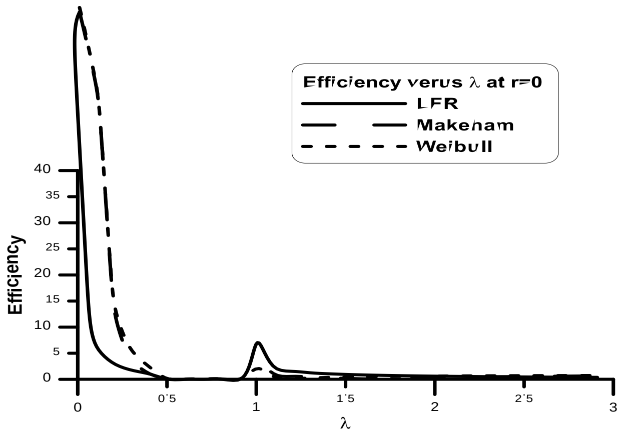

Fig.(1) Efficiency versus λ

From Fig.(1), the maximum efficiency will be at λ = 1.01, then from equations (5.1), (5.2)and (5.3) the efficiency of the three distribution are tabulated in Table-I.

Table-I Pitman Asymptotic efficiency

Distribution	Efficiency
LFR	7.07
Makeham	3.53
Weibull	2.82

6- Monte Carlo Null Distribution Critical Points

In this section a simulation for the null distribution critical points for $\hat{\delta}$ will be made for sample sizes *n=5(1)40, 45* and*50* from the standard exponential distribution. Table-II gives the upper percentile of the statistic $\hat{\delta}$. Fig. (2) shows the relation between the critical values and the sample size.

Table II: Critical values of $\hat{\delta}$

n	90%	95%	98%	99%
5	-50.192	-39.886	-31.352	-26.02
6	-53.074	-43.752	-34.152	-29.611
7	-54.605	-46.201	-37.824	-32.537
8	-58.616	-49.8	-40.493	-35.458
9	-61.494	-52.483	-44.254	-39.611
10	-62.879	-54.027	-47.023	-41.644
11	-64.906	-56.765	-48.766	-44.04
12	-65.742	-58.541	-50.591	-45.291
13	-67.405	-60.365	-52.513	-46.797
14	-68.099	-60.16	-53.098	-49.005
15	-69.819	-62.708	-56.608	-52.38
16	-70.604	-62.593	-55.285	-50.611
17	-71.276	-64.857	-56.741	-51.73
18	-73.687	-66.149	-58.525	-54.204
19	-72.522	-65.468	-59.115	-55.195
20	-73.102	-66.429	-59.434	-55.236
21	-75.579	-70.166	-62.639	-56.546
22	-75.749	-69.969	-63.392	-59.021
23	-75.901	-69.505	-62.839	-59.473
24	-76.048	-70.449	-63.725	-59.554
25	-76.771	-70.775	-63.475	-59.634

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26	-76.951	-70.842	-65.512	-60.694
27	-77.459	-71.15	-65.828	-61.798
28	-79.003	-72.635	-66.477	-63.178
29	-79.539	-73.63	-67.803	-63.54
30	-79.025	-73.452	-66.998	-62.175
31	-79.254	-73.653	-67.728	-63.925
32	-79.632	-73.764	-67.415	-64.91
33	-80.376	-74.443	-68.798	-64.314
34	-80.564	-74.95	-68.972	-66.349
35	-80.991	-75.329	-69.549	-66.024
36	-81.133	-75.481	-69.604	-67.063
37	-81.389	-76.688	-71	-66.73
38	-81.862	-76.817	-71.676	-67.77
39	-82.116	-76.579	-70.97	-67.564
40	-83.127	-77.935	-72.581	-69.418
45	-83.589	-78.485	-73.505	-70.133
50	-84.819	-79.482	-74.725	-71.417

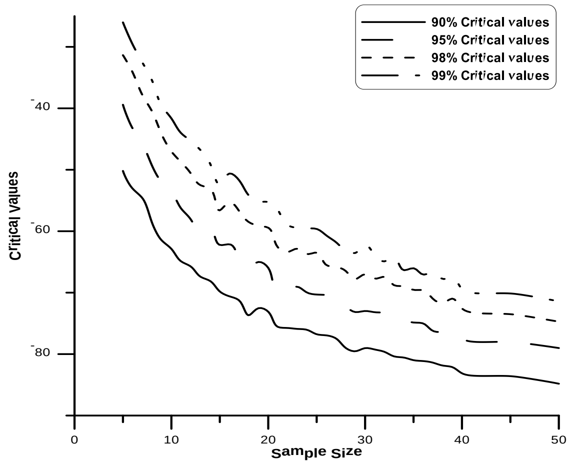

Fig. (2) Relation between critical values and sample size for $\hat{\delta}$

7- Power of the test

In this section an estimation of the power for testing exponentiality versus EBU_{mgf} will be made using significance level 95% with suitable parameters values of θ at *n=10,20 and 30,* and for commonly used distributions in reliability such as LFR, Makeham, and Weibull alternatives. Table-III shows the power of the test.

Table-III: Power estimates

Distribution	**θ**	**n**		
		10	20	30
LFR	2	1	1	1
	3	1	1	1
	4	1	1	1
Makeham	2	0.998	1	1
	3	1	1	1
	4	1	1	1
Weibull	2	0.964	0.969	0.975
	3	0.961	0.970	0.980
	4	0.955	0.971	0.983

8-Example

The following data represent 39 liver cancer's patients taken from El-Minia Cancer Center of Ministry of Health of Egypt, in 1999. The ordered life times (in days) are:

The data are 10, 14, 14, 14, 14, 14, 15, 17, 18, 20, 20, 20, 20, 20, 23, 23, 24, 26, 30, 30, 31, 40, 49, 51, 52, 60, 61, 67, 71, 74, 75, 87, 96, 105, 107, 107, 107, 116, and 150.

It is found that the test statistic for the set of data by using equation (4.2) is $\hat{\delta} = -1.57*10^{64}$ which is greater than the critical value of the Table-II, and then we accept H_1 which states that the set of data have EBU_{mgf} property at 95% percentile.

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