IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 1 Issue 8, October 2014.
www.ijiset.com

ISSN 2348 – 7968

Precise Estimates for The Solution of Stochastic Functional Differential Equations With Discontinuous Initial Data (Part 1)

Tagelsir A Ahmed
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Khartoum, Sudan
E-mail tagelsir6@gmail.com

van Casteren, J.A.
Department of Mathematics and Computer Science,
University of Antwerp (UA), Middelheimlaan 1,
2020 Antwerp, Belgium

August 11, 2014

Abstract

In this work we have used the same introduction,notations and definitions as in [2]. Here we have proved a theorem in which we have established a uniform error bound for the Euler approximation to the solution process of the Stochastic Funtional Differential Equation (S.F.D.E.) (1.11) over the whole time interval $[0, a]$. This Theorem is an extension of the work of Kloeden and Platen ([6], Theorem 10.2.2) to S.F.D.E.'s with discontinuous initial data. We have calculated this uniform error bound by computing the difference between the actual solution process and it's Euler approximation and we have found the upper bound for this difference. We have also discussed the dependence of this difference on the inital data.We have also proved that the Euler approximation of the solution process has the order of strong convergence $\gamma = 0.5$ $see[6]chapters9and10$.

ISSN 2348 – 7968

1 Uniform Error Bound

In the following theorem we shall establish a uniform error bound for the Euler approximation to the solution process of the S.F.D.E. (1.11) over the whole time interval $[0,a]$. This Theorem is an extension of the work of Kloeden and Platen ([6], Theorem 10.2.2) to S.F.D.E.'s with discontinuous initial data.

1 Theorem. *Suppose*

(i) $V \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P; \mathbf{R}^n)$.

(ii) $\theta \in \mathcal{L}^2(J \times \Omega, \mathcal{B}(J) \otimes \mathcal{F}, \lambda \otimes P; \mathbf{R}^n)$.

(iii) $f, g : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to \mathbf{R}^n$ *satisfy*

- *(a)* f *and* g *are* $\mathcal{B}([0,a]) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(L^2(J,\mathbf{R}^n)) - \mathcal{B}(\mathbf{R}^n)$ *measurable.*
- *(b) There exists a constant* K_1 *such that* $\forall t \in [0,a]$; $\forall v \in \mathbf{R}^n$ *and* $\forall h \in \mathcal{L}^2(J,\mathbf{R}^n)$ *we have* $|f(t,v,h)| + |g(t,v,h)| \le K_1(|v| + \|h\| + 1)$.
- *(c) There exists a constant* K_2 *such that for all* $t \in [0,a]$; $\forall (v_i, h_i) \in \mathbf{R}^n \times L^2(J,\mathbf{R}^n)$, $i = 1,2$ *we have* $|f(t,v_1,h_1) - f(t,v_2,h_2)| + |g(t,v_1,h_1) - g(t,v_2,h_2)|$
$\le K_2(|v_1 - v_2| + \|h_1 - h_2\|)$.
- *(d) There exists a constant* K_3 *such that* $\forall t,\ s \in [0,a]$, $t > s$; $\forall (v,h) \in \mathbf{R}^n \times L^2(J,\mathbf{R}^n)$ *we have* $|f(t,v,h) - f(s,v,h)| + |g(t,v,h) - g(s,v,h)| \le K_3(|v| + \|h\| + 1)|t-s|^{1/2}$.

Then if X^π *is the Euler approximation of the solution process of the S.F.D.E. (1.11), we have the following uniform error bound:* $\forall t \in [0,a]$

$$\mathbf{E}\left\{ \sup_{0\le s\le t} \|(X^\pi s), X_s^\pi) - (x(s), x_s)\| \right\} \le K_4\|(V,\theta) - (V',\theta')\| + K_4'\delta^{1/2} \qquad (1.1)$$

where the constants K_1, K_2, K_3, K_4 *and* K_4' *do not depend on* δ.

To prove Theorem 1 we shall need the following remark and lemma.

1.1 Remark

The following two estimates hold:

(a) $\|(x(t), x_t) - (x(s), x_s)\|^2 \le C_1(\|V\|^2 + \|\theta\|^2 + 1)(t-s) \quad \forall 0 \le s \le t \le a$
where C_1 is a constant depending only on K_1 and a.

(b) $\mathbf{E}\left\|(x(t),x_t)\right\|^2 \le \mathbf{E}\left(\sup_{0\le s\le a}\left\|(x(s),x_s)\right\|^2 |\mathcal{F}_0\right) \le S^\star\left(\|V\|^2+\|\theta\|^2+1\right)$ $\forall t\in[0,a]$, where $S^\star$ is the constant of Remark (2.8) in [1].

Proof of Remark. To prove part (a) of this remark let $0\le s\le t\le a$. Then we have

$$\left\|(x(t),x_t)-(x(s),x_s)\right\|^2 = \left\|x(t)-x(s)\right\|^2+\left\|x_t-x_s\right\|^2 \tag{1.2}$$

Also by using the properties of the integrals we have

$$\begin{aligned}
&\left\|x(t)-x(s)\right\|^2\\
&\le 2\left\|\int_s^t f(u,x(u),x_u)du\right\|^2+2\left\|\int_s^t g(u,x(u),x_u)dW(u)\right\|^2\\
&\le 2a\int_s^2 \left\|f(u,x(u),x_u)\right\|^2 du+2\int_s^t \left\|g(u,x(u),x_u)\right\|^2 du\\
&\le 6aK_1^2\int_s^t\left(\|x(u)\|^2+\|x_u\|^2+1\right)du+6K_1^2\int_s^t\left(\|x(u)\|^2+\|x_u\|^2+1\right)du\\
&\quad\text{(by condition (iii)(b) of Theorem 1)}\\
&\le 6K_1^2(a+1)\int_s^t\left(\|(x(u),x_u)\|^2+1\right)du\\
&\le 12K_1^2(a+1)S^\star\left(\|V\|^2+\|\theta\|^2+1\right)\int_s^t du
\end{aligned}$$

(by the estimate in [1])

$$= 12K_1^2(a+1)S^\star\left(\|V\|^2+\|\theta\|^2+1\right)(t-s) \tag{1.3}$$

Now by using (1.3) we also have

$$\begin{aligned}
\left\|x_t-x_s\right\|^2 &= \int_{-1}^0 \left\|x(t+r)-x(s+r)\right\|^2 dr\\
&\le 12K_1^2(a+1)S^\star\left(\|V\|^2+\|\theta\|^2+1\right)(t-s).
\end{aligned} \tag{1.4}$$

Now by inserting (1.3) and (1.4) in (1.2) we get $\forall 0\le s\le t\le a$

$$\left\|(x(t),x_t)-(x(s),x_s)\right\|^2 \le C_1\left(\|V\|^2+\|\theta\|^2+1\right)(t-s) \tag{1.5}$$

where the constant C_1 depends only on K_1 and a. Part (b) of this remark follows directly from remark(2.8) in [1]. □

2 Proposition. *If X^π is the Euler approximation of the S.F.D.E.* **??**, *then the following inequality is true:*

$$\mathbf{E}\left(\sup_{0\le s\le a}\left\|(X^\pi(s),X_s^\pi)\right\|^2\right) \;\le\; C_0\left(\|V\|^2+\|\theta\|^2+1\right),$$

where $C_0\ge 1$ is a constant which does not depend on δ.

ISSN 2348 – 7968

Proof. Let $t \in [0, a]$; then by using condition (iii)(b) of Theorem 1 and applying Lemma 10.8.1 in Kloeden and Platen [6] and applying the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we get

$$\begin{aligned}
&\mathbf{E}\left\{\sup_{0\leq s\leq t} |X^\pi(s)|^2 \mid \mathcal{F}_0\right\} \\
&\leq 3|V|^2 + 3a\int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq t} |f(s, X^\pi(s), X_s^\pi)|^2 \mid \mathcal{F}_0\right\} du \\
&\quad + 3\cdot 4^3 \int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq t} |g(s, X^\pi(s), X_s^\pi)|^2 \mid \mathcal{F}_0\right\} du \\
&\leq 3|V|^2 + 3^2 a K_1^2 \int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq u} (\|X^\pi(s)\|^2 + \|X_s^\pi\|^2 + 1) \mid \mathcal{F}_0\right\} du \\
&\quad + 3^2\cdot 4^3 K_1^2 \int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq u} (\|X^\pi(s)\|^2 + \|X_s^\pi\|^2 + 1) \mid \mathcal{F}_0\right\} du \qquad (1.6) \\
&= 3|V|^2 + 3^2 K_1^2 (a+4^3) \int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq u} (\|X^\pi(s)\|^2 + \|X_s^\pi\|^2 + 1) \mid \mathcal{F}_0\right\} du.
\end{aligned}$$

Here $|V|^2$ is the Euclidean norm of V. It is tacitly understood that V as well as $\theta(s)$, $-1 \leq s \leq 0$ are stochastic variables which are measurable with respect to $\mathcal{F}_0$. We also have, for $t \in [0, a]$,

$$\begin{aligned}
\|X_t^\pi\|^2 &= \int_{-1}^0 |X^\pi(t+s)|^2\, ds = \int_{t-1}^t |X^\pi(s)|^2\, ds \leq \int_{-1}^t |X^\pi(r)|^2\, dr \\
&\leq \int_{-1}^0 |X^\pi(r)|^2\, dr + \int_0^t |X^\pi(r)|^2\, dr = |\theta|^2 + \int_0^t |X^\pi(r)|^2\, dr,
\end{aligned}$$

where $|\theta|^2 = \int_{-1}^0 |\theta(s)|^2\, ds$. Thus we get

$$\left\{\sup_{0\leq s\leq t} \|X_s^\pi\|^2\right\} \leq a\left\{\sup_{0\leq s\leq t} |X^\pi(s)|^2\right\} + |\theta|^2. \qquad (1.7)$$

A combination of (1.6) and (1.7) yields

$$\begin{aligned}
&\mathbf{E}\left\{\sup_{0\leq s\leq t} |X^\pi(s)|^2 \mid \mathcal{F}_0\right\} \\
&\leq 3|V|^2 + 3^2 K_1^2 (a+4^3) \int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq u} ((1+a)|X^\pi(s)|^2 + |\theta|^2 + 1) \mid \mathcal{F}_0\right\} du \\
&\leq 3|V|^2 + 3^2 K_1^2 (a+4^3)(1+|\theta|^2)a \\
&\quad + 3^2 K_1^2 (a+4^3)(1+a) \int_0^t \mathbf{E}\left\{\sup_{0\leq s\leq u} |X^\pi(s)|^2 \mid \mathcal{F}_0\right\} du.
\end{aligned}$$

ISSN 2348 – 7968

So we obtain $\phi(t) \leq \alpha + L\int_0^t \phi(u)du$; where $\phi(t) = \mathbf{E}\left\{\sup_{0\leq s\leq t}|X^\pi(s)|^2 \mid \mathcal{F}_0\right\}$; $\alpha = 3|V|^2 + 3^2K_1^2(a+4^3)(1+|\theta|^2)a$ and $L = 3^2K_1^2(a+4^3)(1+a)$. Now by applying Grönwall‘s lemma we get

$$\phi(t) \leq \alpha\exp(Lt), \quad \forall t\in[0,a].$$

Thus

$$\phi(t) \leq \alpha\exp(La), \quad \forall t\in[0,a],$$

and hence we have

$$\mathbf{E}\left\{\sup_{0\leq s\leq a}|X^\pi(s)|^2 \mid \mathcal{F}_0\right\} \leq C\left(|V|^2+|\theta|^2+1\right), \tag{1.8}$$

where the constant $C = \max\{3\exp La, 3^2K_1^2(a+4^3)a\exp(La)\}$ does not depend on δ. Now by combining (1.7) and (1.8) we get

$$\begin{aligned}\mathbf{E}\left\{\sup_{0\leq t\leq a}\|(X^\pi(t),X_t^\pi)\|^2 \mid \mathcal{F}_0\right\} &\leq \mathbf{E}\left\{\sup_{0\leq t\leq a}|X^\pi(t)|^2 \mid \mathcal{F}_0\right\}\\ &\quad + \mathbf{E}\left\{\sup_{0\leq t\leq a}\|X_t^\pi\|^2 \mid \mathcal{F}_0\right\}\\ &\leq (1+a)\mathbf{E}\left\{\sup_{0\leq t\leq a}|X^\pi(t)|^2 \mid \mathcal{F}_0\right\} + |\theta|^2\\ &\leq (1+a)C\left(|V|^2+|\theta|^2+1\right)+|\theta|^2\\ &\leq C_0\left(|V|^2+|\theta|^2+1\right), \end{aligned} \tag{1.9}$$

where the constant $C_0 = \max\{1,(1+a)C\}$ depends only on K_1 and a and hence does not depend on δ. Upon taking expectations in (1.9) we get

$$\mathbf{E}\left\{\sup_{0\leq t\leq a}\|(X^\pi(t),X_t^\pi)\|^2\right\} \leq C_0\left(|V|^2+|\theta|^2+1\right). \tag{1.10}$$

□

Proof of Theorem (2.1). Here X^π denotes the Euler approximation of the solution of the S.F.D.E.

$$x(t) = \begin{cases} V + \int_0^t f(u,x(u),x_u)du + \int_0^t g(u,x(u),x_u)dW(u) & 0\leq t\leq a\\ \theta(t) & t\in J\end{cases} \tag{1.11}$$

where $X^\pi(0) = V'$ and $X_0^\pi(s) = \theta'(s) \quad s\in J = [-1,0)$ and $V'\in\mathcal{L}^2(\Omega,\mathcal{F}_0,P;\mathbf{R}^n)$ and $\theta'\in\mathcal{L}^2(J\times\Omega,\mathcal{H}(J)\otimes\mathcal{F}_0,\lambda\otimes P;\mathbf{R}^n)$ and for $t_n < t\leq t_{n+1}$ we have

$$X^\pi(t) \;=\; X^\pi(t_n) + \int_{t_n}^t f\left(t_n, X^\pi(t_n), X_{t_n}^\pi\right)du + \int_{t_n}^t g\left(u, X^\pi(u), X_u^\pi\right)dW(u)$$

and $\delta = \max\{(t_{i+1} - t_i) : i = 1, 2, \cdots, m\}$ is such that $\delta \in (0, 1)$.
Note: Please observe that there is no necessary relation between the n used in t_n and the n used in $\mathbf{R}^n$.
Thus for $t_n < t \leq t_{n+1}$ we can rewrite X^π as follows:

$$\begin{aligned} X^\pi(t) &= V' + \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} f\left(t_n, X^\pi(t_n), X^\pi_{t_n}\right) du + \int_{t_{n_t}}^{t} f\left(t_{n_t}, X^\pi(t_{n_t}), X^\pi_{t_{n_t}}\right) du \\ &+ \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} g\left(t_n, X^\pi(t_n), X^\pi_{t_n}\right) dW(u) \\ &+ \int_{t_{n_t}}^{t} g\left(t_{n_t}, X^\pi(t_{n_t}), X^\pi_{t_{n_t}}\right) dW(u). \end{aligned} \tag{1.12}$$

where $n_t = \max\{n : n \in \{0, 1, \cdots, m\}$ and $t_n \leq t\}$.
Also for $t_n < t \leq t_{n+1}$ we can rewrite $x(t)$ as follows:

$$\begin{aligned} x(t) &= V + \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} f\left(u, x(u), x_u\right) du + \int_{t_{n_t}}^{t} f\left(u, x(u), x_u\right) du \\ &+ \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} g\left(u, x(u), x_u\right) dW(u) + \int_{t_{n_t}}^{t} g\left(u, x(u), x_u\right) dW(u). \end{aligned} \tag{1.13}$$

Now by equations (1.12) and (1.13) we have

$$\begin{aligned} x(t) - X^\pi(t) &= V - V' + \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} \left(f(u, x(u), x_u) - f(t_n, X^\pi(t_n), X^\pi_{t_n})\right) du \\ &+ \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} \left(g(u, x(u), x_u) - g(t_n, X^\pi(t_n), X^\pi_{t_n})\right) dW(u) \\ &+ \int_{t_{n_t}}^{t} \left(f(u, x(u), x_u) - f(t_{n_t}, X^\pi(t_{n_t}), X^\pi_{t_{n_t}})\right) du \\ &+ \int_{t_{n_t}}^{t} \left(g(u, x(u), x_u) - g(t_{n_t}, X^\pi(t_{n_t}), X^\pi_{t_{n_t}})\right) dW(u) \\ &= V - V' + \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} \left(f(u, x(u), x_u) - f(t_n, x(u), x_u)\right) du \\ &+ \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} \left(f(t_n, x(u), x_u) - f(t_n, x(t_n), x_{t_n})\right) du \\ &+ \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} \left(f(t_n, x(t_n), x_{t_n}) - f(t_n, X^\pi(t_n), X^\pi_{t_n})\right) du \\ &+ \sum_{n=1}^{n_t-1} \int_{t_n}^{t_{n+1}} \left(g(u, x(u), x_u) - g(t_n, x(u), x_u)\right) dW(u) \end{aligned}$$

ISSN 2348 – 7968

$$
\begin{aligned}
&+\sum_{n=1}^{n_t-1}\int_{t_n}^{t_{n+1}} \left(g(t_n, x(u), x_u) - g(t_n, x(t_n), x_{t_n})\right) dW(u)\\
&+\sum_{n=1}^{n_t-1}\int_{t_n}^{t_{n+1}} \left(g(t_n, x(t_n), x_{t_n}) - g(t_n, X^\pi(t_n), X^\pi_{t_n})\right) dW(u)\\
&+\int_{t_{n_t}}^{t} \left(f(u, x(u), x_u) - f(t_{n_t}, x(u), x_u)\right) du\\
&+\int_{t_{n_t}}^{t} \left(f(t_{n_t}, x(u), x_u) - f(t_{n_t}, x(t_{n_t}), x_{t_{n_t}})\right) du\\
&+\int_{t_{n_t}}^{t} \left(f(t_{n_t}, x(t_{n_t}), x_{t_{n_t}}) - (t_{n_t}, X^\pi(t_{n_t}), X^\pi_{t_{n_t}})\right) du\\
&+\int_{t_{n_t}}^{t} \left(g(u, x(u), x_u) - g(t_{n_t}, x(u), x_u)\right) dW(u)\\
&+\int_{t_{n_t}}^{t} \left(g(t_{n_t}, x(u), x_u) - g(t_{n_t}, x(t_{n_t}), x_{t_{n_t}})\right) dW(u)\\
&+\int_{t_{n_t}}^{t} \left(g(t_{n_t}, x(t_{n_t}), x_{t_{n_t}}) - g(t_{n_t}, X^\pi(t_{n_t}), X^\pi_{t_{n_t}})\right) dW(u) \qquad (1.14)
\end{aligned}
$$

Now let

$$Z(t) = \mathbf{E}\left\{sup_{0\le s\le t}\left\|(x(s), x_s) - (X^\pi(s), X^\pi_s)\right\|^2 |\mathcal{F}_0\right\} \qquad (1.15)$$

Now by using the definition of x_s and X^π_s we have for each $s \in [0, a]$

$$
\begin{aligned}
\|x_s - X^\pi_s\|^2 &= \int_{-1}^{0} |x(s+r) - X^\pi(s+r)|^2 dr\\
&\le \int_{-1}^{s} |x(u) - X^\pi(u)|^2 du\\
&\le a \sup_{0\le u\le s} |x(u) - X^\pi(u)|^2 + \int_{-1}^{0} |\theta(s) - \theta'(s)|^2 ds
\end{aligned}
$$

Now by taking the supremum over $s \in [0, t]$ and the conditional expectation in both sides of the above inequality, we get for each $t \in [0, a]$

$$\mathbf{E}\left\{\sup_{0\le s\le t}\|x_s - X^\pi_s\|^2 \,\middle|\, \mathcal{F}_0\right\} \le a\mathbf{E}\left\{\sup_{0\le s\le t}|x(s) - X^\pi(s)|^2 \,\middle|\, \mathcal{F}_0\right\} + |\theta - \theta'|^2 \qquad (1.16)$$

Now by combining (1.15) and (1.16) we get

$$Z(t) \le (1+a)\mathbf{E}\left\{\sup_{0\le s\le t}|x(s) - X^\pi(s)|^2|\mathcal{F}_0\right\} + |\theta - \theta'|^2$$

ISSN 2348 – 7968

where we denote

$$|(V,\theta)-(V',\theta')|^2 = |V-V'|^2 + |\theta-\theta'|^2 = |V-V'|^2 + \int_{-1}^{0} |\theta(s)-\theta'(s)|^2\, ds$$

Then by using equation (1.14) and the inequality $\left(\sum_{i=1}^{7} a_i\right)^2 \leq 7\sum_{i=1}^{7} a_i^2$ we get

$$Z(t) \leq 7(a+1)\,|(V,\theta)-(V',\theta')|^2 + \sum_{j=0}^{1} 7(1+a)(R_t^{(j)} + S_t^{(j)} + T_t^{(j)}) \tag{1.17}$$

where

$$R_t^{(0)} = \mathbf{E}\left\{ \sup_{0\leq s\leq t} \left| \sum_{n=1}^{n_s-1} \int_{t_n}^{t_{n+1}} \big(f(t_n, x(t_n), x_{t_n}) - f(t_n, X^\pi(t_n), X^\pi_{t_n})\big)\, du + \int_{t_{n_s}}^{s} \big(f(t_{n_s}, x(t_{n_s}), x_{t_{n_s}}) - f(t_{n_s}, X^\pi(t_{n_s}), X^\pi_{t_{n_s}})\big)\, du \right|^2 |\mathcal{F}_0 \right\}; \tag{1.18}$$

$$R_t^{(1)} = \mathbf{E}\left\{ \sup_{0\leq s\leq t} \left| \sum_{n=1}^{n_s-1} \int_{t_n}^{t_{n+1}} \big(g(t_n, x(t_n), x_{t_n}) - g(t_n, X^\pi(t_n), X^\pi_{t_n})\big)\, dW(u) + \int_{t_{n_s}}^{s} \big(g(t_{n_s}, x(t_{n_s}), x_{t_{n_s}}) - g(t_{n_s}, X^\pi(t_{n_s}), X^\pi_{t_{n_s}})\big)\, dW(u) \right|^2 |\mathcal{F}_0 \right\}; \tag{1.19}$$

$$S_t^{(0)} = \mathbf{E}\left\{ \sup_{0\leq s\leq t} \left| \sum_{n=1}^{n_s-1} \int_{t_n}^{t_{n+1}} (f(t_n, x(u), x_u) - f(t_n, x(t_n), x_{t_n}))\, du + \int_{t_{n_s}}^{s} \big(f(t_{n_s}, x(u), x_u) - f(t_{n_s}, x(t_{n_s}), x_{t_{n_s}})\big)\, du \right|^2 |\mathcal{F}_0 \right\}; \tag{1.20}$$

$$S_t^{(1)} = \mathbf{E}\left\{ \sup_{0\leq s\leq t} \left| \sum_{n=1}^{n_s-1} \int_{t_n}^{t_{n+1}} (g(t_n, x(u), x_u) - g(t_n, x(t_n), x_{t_n}))\, dW(u) + \int_{t_{n_s}}^{s} \big(g(t_{n_s}, x(u), x_u) - g(t_{n_s}, x(t_{n_s}), x_{t_{n_s}})\big)\, dW(u) \right|^2 |\mathcal{F}_0 \right\}; \tag{1.21}$$

$$T_t^{(0)} = \mathbf{E}\left\{ \sup_{0\leq s\leq t} \left| \sum_{n=1}^{n_s-1} \int_{t_n}^{t_{n+1}} (f(u, x(u), x_u) - f(t_n, x(u), x_u))\, du + \int_{t_{n_s}}^{s} (f(u, x(u), x_u) - f(t_{n_s}, x(u), x_u))\, du \right|^2 |\mathcal{F}_0 \right\}; \tag{1.22}$$

$$T_t^{(1)} = \mathbf{E}\left\{ \sup_{0\leq s\leq t} \left| \sum_{n=1}^{n_s-1} \int_{t_n}^{t_{n+1}} (g(u, x(u), x_u) - g(t_n, x(u), x_u))\, dW(u) \right.\right.$$

ISSN 2348 – 7968

$$+\int_{t_{n_s}}^{s}\left(g(u,x(u),x_u)-g(t_{n_s},x(u),x_u)\right)dW(u)\Big|^2|\mathcal{F}_0\Big\}. \tag{1.23}$$

Observe that $R_t^{(j)}$, $S_t^{(j)}$ and $T_t^{(j)}$ are finite for $j=1,2$ (because of inequality (1.9) and part (b) of Remark (1.1) and hence we can apply Lemma 10.8.1 of [6] to $R_t^{(j)}$, $S_t^{(j)}$ and $T_t^{(j)}$, $j=1,\ 2$.

Now applying Lemma (10.8.1) (in Kloeden and Platen [6]) to $R_t^{(1)}$ in (1.19) and using condition(iii)(c) of Theorem 1 we get

$$\begin{aligned}
R_t^{(1)} &\le 4^3\int_0^t \mathbf{E}\left\{\sup_{0\le s\le u}\left|g(t_{n_s},x(t_{n_s}),x_{t_{n_s}})-g(t_{n_s},X^\pi(t_{n_s},X^\pi_{t_{n_s}})\right|^2\,\Big|\,\mathcal{F}_0\right\}du\\
&\le 4^3\cdot 2K_2^2\int_0^t \mathbf{E}\left\{\sup_{0\le s\le u}\left(|x(t_{n_s})-X^\pi(t_{n_s})|^2+\|x_{t_{n_s}}-X^\pi_{t_{n_s}}\|^2\right)|\mathcal{F}_0\right\}du\\
&\le 4^3\cdot 2K_2^2\int_0^t \mathbf{E}\left\{\sup_{0\le s\le u}\left(|x(s)-X^\pi(s)|^2+\|x_s-X^\pi_s\|^2\right)|\mathcal{F}_0\right\}du\\
&= 4^3\cdot 2K_2^2\int_0^t Z(u)du
\end{aligned} \tag{1.24}$$

Now by applying Lemma (10.8.1) (in Kloeden and Platen [6]) to $R_t^{(0)}$ in (1.18) and using an argument analogous to that in (1.24) we get

$$R_t^{(0)} \le 2aK_2^2\int_0^t Z(u)du \tag{1.25}$$

Now (1.24) and (1.25) can be written as

$$R_t^{(j)} \le C_2K_2^2\int_0^t Z(u)du \quad \text{(j= 0,1)} \tag{1.26}$$

where the constant C_2 does not depend on δ.

Furthermore by applying lemma(10.8.1)(in [6]) to $S_t^{(1)}$ in (1.21) and using part(a) of remark 1.1 and condition(iii)(c)of Theorem 1 we obtain

$$\begin{aligned}
S_t^{(1)} &\le 4^3\cdot 2K_2^2\int_0^t \mathbf{E}\left\{sup_{0\le s\le u}\left\|(x(t_{n_s}),x_{t_{n_s}})-(x(s),x_s)\right\|^2|\mathcal{F}_0\right\}du\\
&\le 4^3\cdot 2K_2^2\int_0^t \mathbf{E}\left\{sup_{0\le s\le u}C_1(|V|^2+|\theta|^2+1)(s-t_{n_s})\right\}du\\
&\le 4^3\cdot 2K_2^2aC_1(|V|^2+|\theta|^2+1)\delta
\end{aligned} \tag{1.27}$$

Now by applying Lemma 10.8.1 (in [6]) to $S_t^{(0)}$ in (1.20) and using an argument similar to that in (1.27) we get

$$S_t^{(0)} \le 2K_2^2a^2C_1(|V|^2+|\theta|^2+1)\delta \tag{1.28}$$

Then by combining (1.27) and (1.28) we get

$$S_t^{(j)} \;\le\; C_3(|V|^2+|\theta|^2+1)\delta \quad \text{(j=0,1)} \tag{1.29}$$

where C_3 is a constant independent of δ.
Similarly applying condition (iii)(d) of Theorem 1 and Lemma (10.8.1) (in [6]) and (part(b)of remark (2.8) in [1]) we get

$$\begin{aligned} T_t^{(1)} &\le 4^3\int_0^t \mathbf{E}\left\{\sup_{0\le s\le u}|g(s,x(s),x_s)-g(t_{n_s},x(s),x_s)|^2\,|\mathcal{F}_0\right\}du \\ &\le 4^3\cdot 3K_3^2\int_0^t \mathbf{E}\left\{\sup_{0\le s\le u}\left((\|(x(s),x_s)\|^2+1)(s-t_{n_s})\right)\right\}du \\ &\le 4^3\cdot 3K_3^2\delta\int_0^t \left(S^\star(|V|^2+|\theta|^2+1)+1\right)du \\ &\le 4^3\cdot 6K_3^2\delta\int_0^t S^\star(|V|^2+|\theta|^2+1)du \quad (\text{as } S^\star\ge 1) \\ &\le 4^3\cdot 6K_3^2 aS^\star(|V|^2+|\theta|^2+1)\delta \end{aligned} \tag{1.30}$$

Now by applying Lemma 10.8.1 (in [6]) to $T_t^{(0)}$ in (1.22) and using an argument similar to that in (1.30) we get

$$T_t^{(0)} \le 6K_3^2a^2S^\star\left(|V|^2+|\theta|^2+1\right)\delta. \tag{1.31}$$

Now by combining (1.30) and (1.31) we get

$$T_t^{(j)} \le C_4(|V|^2+|\theta|^2+1)\delta \quad \text{(j=0,1)}, \tag{1.32}$$

where the constant C_4 does not depend on δ.
Now by inequalities (1.17), (1.26), (1.29) and (1.32) we get

$$\begin{aligned} Z(t) &\le 7(a+1)|(V,\theta)-(V',\theta')|^2 \\ &\quad +14(1+a)\left\{C_2K_2^2\int_0^t Z(u)du+(C_3+C_4)(|V|^2+|\theta|^2+1)\delta\right\} \end{aligned}$$

Hence

$$\begin{aligned} Z(t) &\le 7(a+1)\,|(V,\theta)-(V',\theta')|^2 \\ &\quad +C_5(|V|^2+|\theta|^2+1)\delta+C_6\int_0^t Z(u)du, \end{aligned} \tag{1.33}$$

where the constants C_5 and C_6 are independent of δ
Now by applying Grönwall's inequality to (1.33) we get, $\forall t\in[0,a]$,

$$Z(t) \;\le\; \left\{7(a+1)\,|(V,\theta)-(V',\theta')|^2+C_5(|V|^2+|\theta|^2+1)\delta\right\}\exp\left(C_6t\right)$$

$$\leq \left\{7(a+1)\left|(V,\theta)-(V',\theta')\right|^2 + C_5(|V|^2+|\theta|^2+1)\delta\right\}\exp(C_6 a)$$

Hence $\forall t \in [0,a]$ we have

$$Z(t) \leq C_7\left\{\left|(V,\theta)-(V',\theta')\right|^2 + \left(|V|^2+|\theta|^2+1\right)\delta\right\}, \tag{1.34}$$

where the constant $C_7 = \max\{C_5\exp(C_6 a), 7(1+a)\exp(C_6 a)\}$ does not depend on δ. Now by denoting $Y(t) = \sup_{0\leq s\leq t}\|(x(s),x_s)-(X^\pi(s),X^\pi_s)\|$ and applying Hölder inequality and using inequality (1.34) and the properties of the conditional expectation we get

$$\begin{aligned}
&\left(\mathbf{E}\left\{\sup_{0\leq s\leq t}\|(x(s),x_s)-(X^\pi(s),X^\pi_s)\|\right\}\right)^2 \\
&= (\mathbf{E}Y(t))^2 \leq \mathbf{E}Y^2(t) = \mathbf{E}\left\{\mathbf{E}\{Y^2(t)|\mathcal{F}_0\}\right\} = \mathbf{E}\{Z(t)\} \\
&\leq C_7\mathbf{E}\left\{|(V,\theta)-(V',\theta')|^2 + (|V|^2+|\theta|^2+1)\delta\right\} \\
&= C_7\left\{\|(V,\theta)-(V',\theta')\|^2 + (\|V\|^2+\|\theta\|^2+1)\delta\right\} \\
&\leq C_7\left(\|(V,\theta)-(V',\theta')\| + (\|V\|^2+\|\theta\|^2+1)^{1/2}\delta^{1/2}\right)^2.
\end{aligned} \tag{1.35}$$

Hence we reach the conclusion of Theorem 1, namely $\forall t \in [0,a]$ we have

$$\mathbf{E}\left\{\sup_{0\leq s\leq t}\|(X^\pi(s),X^\pi_s)-(x(s),x_s)\|\right\} \leq K_4\|(V,\theta)-(V',\theta')\| + K_4'\delta^{1/2}$$

where the constants $K_4 = C_7^{1/2}$ and $K_4' = C_7^{1/2}(\|v\|^2+\|\theta\|^2+1)^{1/2}$ are independent of δ. □

3 Corollary. *If we assume that $\|(V,\theta)-(V',\theta')\| \leq K_5\delta^{1/2}$ in addition to the assumptions (i), (ii) and (iii) of Theorem 1,then the Euler approximation X^π satisfies*

$$\|(X^\pi(a),X^\pi_a)-(x(a)x_a)\| \leq K_6\delta^{1/2} \tag{1.36}$$

where the constants K_5 and K_6 are independent of δ.

In other words the Euler approximation X^π has the order of strong convergence $\gamma = 0.5$ (see chapters 9 and 10 of [6]) .

Proof. Since all the assumptions of Theorem 1 are satisfied then its conclusion also holds and hence we have

$$\begin{aligned}
\mathbf{E}\{\|(X^\pi(a),X^\pi_a)-(x(a)x_a)\|\} &\leq \mathbf{E}\left(\sup_{0\leq s\leq t}\|(X^\pi(s),X^\pi_s)-(x(s),x_s)\|\right) \\
&\leq K_4\|(V,\theta)-(V',\theta')\| + K_4'\delta^{1/2} \leq K_6\delta^{1/2},
\end{aligned}$$

where the constant $K_6 = \max\{K_4,K_5,K_4'\}$ does not depend on δ. □

ISSN 2348 – 7968

1.2 Remarks:

(a) All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z : [0,a] \times \Omega \to \mathbf{R}$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities

$$\begin{aligned} |\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| &\le K(t-s) \quad \text{and} \\ \mathbf{E}\left(|Z(t) - Z(s)|^2 \,|\mathcal{F}_s\right) &\le K(t-s) \quad \text{for } 0 \le s \le t \le a. \end{aligned}$$

Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work.

(b) Theorem 1 and Corollary 3 can be extended to a processes $f', g' : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to L(\mathbf{R}^m, \mathbf{R}^n) \quad (m, n \in \mathbf{N})$ instead of the processes $f, g : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to \mathbf{R}^n \quad (n \in \mathbf{N})$, and instead of the Brownian motion W we use the process $Z : [0,a] \times \Omega \to \mathbf{R}^m$ which is a martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$, continuous on $[0,a]$, and has independent increments and satisfies for some constant K the inequalities

$$|\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| \le K(t-s) \quad \text{and} \quad \mathbf{E}\left(|Z(t) - Z(s)|^2 \,|\mathcal{F}_s\right) \le K(t-s)$$

for $0 \le s \le t \le a$.

(c) All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0) \quad (r \ge 0)$.

References

[1] Tagelsir A Ahmed, *Stochastic Functional Differential Equations with Discontinuous Initial Data, M.Sc. Thesis, University of Khartoum, Khartoum, Sudan,* (1983).

[2] Tagelsir A Ahmed and Van Casteren, J.A., *Approximation Theorems for The Solution of Stochastic Functional Differential Equations with Discontinuous Initial Data. International Journal of Innovation in Science and Mathematic IJISM*, Volume-2, Issue-2, (March-2014).

[3] Friedman, A., *Stochastic Differential Equations and Applications,* Academic press (1975).

[4] Halmos, P.R., *Measure Theory,* D. Van Nostrand Company (1950).

[5] Ikeda, N. and Watanabe, S, *Stochastic differential equations and Diffusion Processes,* Amsterdam: North-Holland 1981.

[6] Kloeden, P.E and Platen,E., *Numerical Solution of Stochastic Differential Equations,* Springer-Verlag Berlin Heidelberg (1992).

[7] McShane, E.J., *Stochastic calculus and Stochastic Models,* Academic Press (1974).

[8] Mohammed, S.E.A., *Stochastic Functional Differential Equations,* Research Notes in Mathematics; Pitman Books Ltd., London (1984).