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Precise Estimates for The Solution of Stochastic Functional Differential Equations With Discontinuous Initial Data(Part 2)

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Abstract

This work is a continuation to the work on Precise Estimates in [3] and the work on Approximation Theorems in [2]. Here we have proved that the Euler approximation of the S.F.D.E. considered in [2] and [3] is in fact numerically stable and weakly consistent.Note that here we have used the same introduction, notations and definitions as in[2] and [3].

0.1 Numerical stability

Here we shall define what is meant by $\mathbf{L}^2$-numerical stability and we shall show that the Euler approximation X^π of the solution x of the S.F.D.E. (1.11) in [3] is $\mathbf{L}^2$-numerically stable.

1 Definition. Let X^π be the Euler approximation of the solution x of the S.F.D.E (1.11) in [3] with $X^\pi(0) = x(0) = V$ and $X^\pi(s) = x(s) = \theta(s) \quad \forall s \in [-1, 0)$. Also

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let $\bar{x}$ be the unique solution of the S.F.D.E. (1.11) in [3] obtained by replacing (V, θ) by $(\bar{V}, \bar{\theta})$ where $\bar{V} \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P; \mathbf{R}^n)$ and $\bar{\theta} \in \mathcal{L}^2(J \times \Omega, \mathcal{H}(J) \otimes \mathcal{F}_0, \lambda \otimes P; \mathbf{R}^n)$. Let $\bar{X}$ be the Euler approximation of the solution $\bar{x}$ with $\bar{X}(0) = \bar{x}(0) = \bar{V}$ and $\bar{X}(s) = \bar{x}(s) = \bar{\theta}(s) \quad \forall s \in [-1, 0)$. Then we say that the approximation $X^\pi(t)$ is $\mathbf{L}^2$-numerically stable if $\mathbf{E}\left\{\|(V, \theta) - (\bar{V}, \bar{\theta})\|^2\right\} \to 0$ as $\delta \to 0$ implies $\sup_{0 \le t \le a} \mathbf{E}\left\{\|(X^\pi(t), X_t^\pi) - (\bar{X}(t), \bar{X}_t)\|^2\right\} \to 0$ as $\delta \to 0$.

2 Proposition. *Using the same notation and settings in the above definition the Euler approximation $X^\pi(t)$ is $\mathbf{L}^2$-numerically stable.*

Proof. It is easy to see that

$$\begin{aligned} &\mathbf{E}\left\{|X^\pi(t) - \bar{X}(t)|^2\right\} \\ &\le 3\mathbf{E}\{|X^\pi(t) - x(t)|^2\} + 3\mathbf{E}\{|x(t) - \bar{x}(t)|^2\} + 3\mathbf{E}\left\{|\bar{X}(t) - \bar{x}(t)|^2\right\} \end{aligned} \tag{0.1}$$

and

$$\mathbf{E}\left\{\|X_t^\pi - \bar{X}_t\|^2\right\} \le 3\mathbf{E}\left\{\|X_t^\pi - x_t\|^2\right\} + 3\mathbf{E}\left\{\|x_t - \bar{x}_t\|^2\right\} + 3\mathbf{E}\left\{\|\bar{X}_t - \bar{x}_t\|^2\right\} \tag{0.2}$$

Now by combining (0.1) and (0.2) and using ([**?**]Theorem (2.1) and inequality (1.35)) we get for each $t \in [0, a]$

$$\begin{aligned} &\mathbf{E}\{\|(X^\pi(t), X_t^\pi) - (\bar{X}(t), \bar{X}_t)\|^2\} \\ &\le 3\mathbf{E}\{\|(X^\pi(t), X_t^\pi) - (x(t), x_t)\|^2\} + 3\mathbf{E}\{\|(x(t), x_t) - (\bar{x}(t), \bar{x}_t)\|^2\} \\ &\quad + 3\mathbf{E}\{\|(\bar{X}(t), \bar{X}_t) - (\bar{x}(t), \bar{x}_t)\|^2\} \\ &\le K_7\delta + K_8\mathbf{E}\{\|(V, \theta) - (\bar{V}, \bar{\theta})\|^2\} + K_9\delta, \end{aligned} \tag{0.3}$$

where K_7, K_8 and K_9 are constants independent of δ.
Now since (0.3) holds $\forall t \in [0, a]$, we have

$$\begin{aligned} &\sup_{0 \le t \le a} \mathbf{E}\left\{\|(X^\pi(t), X_t^\pi) - (\bar{X}(t), \bar{X}_t)\|^2\right\} \\ &\le (K_7 + K_9)\delta + K_8\mathbf{E}\left\{\|(V, \theta) - \bar{V}, \bar{\theta})\|^2\right\}. \end{aligned} \tag{0.4}$$

Now suppose that $\mathbf{E}\{\|(V, \theta) - \bar{V}, \bar{\theta})\|^2\} \to 0$ as $\delta \to 0$, then by (0.4) it is easy to see that $\sup\limits_{0 \le t \le a} \mathbf{E}\left\{\|(X^\pi(t), X_t^\pi) - (\bar{X}(t), \bar{X}_t)\|^2\right\} \to 0$ as $\delta \to 0$. Hence the Euler approximation $X^\pi(t)$ is $\mathbf{L}^2$-numerically stable. □

0.2 Weak Consistency

We say that the Euler approximation X^π of the solution of the S.F.D.E (1.11) in [?]with maximum step size δ is weakly consistent if there exist a nonnegative function $C(\delta)$ with

$$\lim_{\delta \to 0^+} C(\delta) = 0 \tag{0.5}$$

such that:

$$\mathbf{E}\left\{\left|\mathbf{E}\left(\frac{X^\pi(t_{k+1})-X^\pi(t_k)}{\Delta_k}|\mathcal{F}_{t_k}\right)-f(t_k,X^\pi(t_k),X^\pi_{t_k})\right|^2\right\}\leq C(\delta) \tag{0.6}$$

and

$$\mathbf{E}\left\{\left|\mathbf{E}\left(\frac{1}{\Delta_k}\left(X^\pi(t_{k+1})-X^\pi\left(t_k\right)\right)\left(X^\pi(t_{k+1})-X^\pi(t_k)\right)^T \mid \mathcal{F}_{t_k}\right)\right.\right.$$
$$\left.\left.-\,g\left(t_k,X^\pi(t_k),X^\pi_{t_k}\right)g\left(t_k,X^\pi(t_k),X^\pi_{t_k}\right)^T\right|^2\right\}\leq C(\delta) \tag{0.7}$$

for all fixed values of $X^\pi(t_k)$ and $k=1,2,\cdots,m$.

3 Proposition. *The Euler approximation X^π of the solution process of the S.F.D.E. (1.11) in [?] is weakly consistent. In other words X^π satisfies (0.5), (0.6) and (0.7).*

To prove Proposition 3 we need the following lemma which we have established by suitable modifications of Theorem 4.5.4 in [7]

4 Lemma. *If X^π is the Euler approximation of the solution of the S.F.D.E. (1.11)in [?], then*

$$\sup_{0\leq u\leq t}\mathbf{E}\left\{\|(X^\pi(u),X^\pi_u)\|^{2p}\right\}\leq C_1\left(\|V\|^{2p}+\|\theta\|^{2p}+1\right)e^{C_1t}, \tag{0.8}$$

where p is any positive integer and C_1 is a constant depending only on K, a and p and not on δ.

Proof. Consider the S.F.D.E.

$$\begin{aligned} X^\pi(t) &= V+\int_0^t f^\pi(u)du+\int_0^t g^\pi(u)dW(u) \quad \text{if } 0\leq t\leq a\\ X^\pi(t) &= \theta(t) \quad \text{if } -1\leq t<0, \end{aligned} \tag{0.9}$$

where for each $u\in[0,a]$,

$$f^\pi(u)=f\left(t_k,X^\pi(t_k),X^\pi_{t_k}\right)=f_k(u), \quad \text{and} \quad g^\pi(u)=g\left(t_k,X^\pi(t_k),X^\pi_{t_k}\right)=g_k(u)$$

for some $k\in\mathbf{N}$, such that $1\leq k\leq m$, and $t_k<u\leq t_{k+1}$. Equivalently the above S.F.D.E. can be written as

$$\begin{aligned} dX^\pi(t) &= f^\pi(t)dt+g^\pi(t)dW(t) \quad \text{if } 0\leq t\leq a\\ X^\pi(t) &= \theta(t) \quad \text{if } -1\leq t<0. \end{aligned} \tag{0.10}$$

Now let $t \in [0, a]$, then by Itô formula we get

$$\begin{aligned} |X^\pi(t)|^{2p} &= |V|^{2p} + \int_0^t 2p|X^\pi|^{2p-2}X^\pi(s)f^\pi(s)ds \\ &+ \int_0^t p(2p-1)|X^\pi(s)|^{2p-2}(g^\pi(s))^2 ds \\ &+ \int_0^t 2p|X^\pi(s))|^{2p-2}X^\pi(s)g^\pi(s)dW(s). \end{aligned} \tag{0.11}$$

Now since $\psi(s) = 2p|X^\pi(s)|^{2p-2}X^\pi(s)g^\pi(s) \in \mathbf{L}^2_\omega[0, a]$ fore all $s \in [0, a]$, then by properties of the Ito integral we have $\mathbf{E}\int_0^t \psi(s)dW(s) = 0$ for each $t \in [0, a]$. Now by taking the expectation on both sides of (0.11) and using the definition of f^π and g^π and the linear growth condition on f and g and the continuity of $X^\pi(s)$ and X^π_s for all $s \in [0, a]$ and the inequality $b^{(2p-2)}(b^2 + 1) \le 1 + 2b^{2p}$ we find that

$$\begin{aligned} &\mathbf{E}|X^\pi(t)|^{2p} \\ &= \|V\|^{2p} + \mathbf{E}\int_0^t 2p|X^\pi(s)|^{2p-2}X^\pi(s)f^\pi(s)ds \\ &\quad + \mathbf{E}\int_0^t p(2p-1)|X^\pi(s)|^{2p-2}(g^\pi(s))^2 ds \\ &\le \|V\|^{2p} + \int_0^t 2p \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}|X^\pi(s)|K\left(|X^\pi(s)| + \|X^\pi_s\| + 1\right) du \\ &\quad + \int_0^t 3p(2p-1) \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}K^2\left(|X^\pi(s)|^2 + \|X^\pi_s\|^2 + 1\right) du \\ &\le \|V\|^{2p} + \int_0^t 2p \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}K\left(|X^\pi(s)|^2 + \|X^\pi_s\|^2 + 1\right)^2 du \\ &\quad + 3\int_0^t p(2p-1) \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{(2p-2)}K^2\left(|X^\pi(s)|^2 + \|X^\pi_s\|^2 + 1\right) du \\ &\le \|V\|^{2p} + 3p(2p+1)(K+K^2)\int_0^t \sup_{0\le s\le u} \mathbf{E}\,|X^\pi(s)|^{2p-2}\left(|X^\pi(s)|^2 + \|X^\pi_s\|^2 + 1\right) du \\ &\le \|V\|^{2p} + 6p(2p+1)(K+K^2)\int_0^t \sup_{0\le s\le u} (\mathbf{E}|X^\pi(s)|^{2p} + 1)du \\ &\quad + 3p(2p+1)(K+K^2)\int_0^t \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}\|X^\pi_s\|^2 du. \end{aligned} \tag{0.12}$$

Now we shall show in steps that inequality (0.12) is also true if we replace the left hand side by $\sup_{0\le s\le t}\mathbf{E}|X^\pi(s)|^{2p}$ which is equal to $\mathbf{E}|X^\pi(s_0)|^{2p}$ for some $s_0 \in [0, a]$. In other words we want to prove the following inequality

$$\sup_{0\le s\le t} \mathbf{E}|X^\pi(s)|^{2p} \le \|V\|^{2p} + 6p(2p+1)(K+K^2)\int_0^t \sup_{0\le s\le u} \mathbf{E}(|X^\pi(s)|^{2p} + 1)du$$

$$+ 3p(2p+1)(K+K^2)\int_0^t \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}\|X_s^\pi\|^2 du. \quad (0.13)$$

Step 1: we know that inequality (0.12) is true with t replaced by s_0 in both sides of the inequality.

Step 2: Now step 1 is also true if we replace s_0 in the right hand side by t and leave s_0 in the left hand side as it is .

Step 3: Now replace $\mathbf{E}|X^\pi(s_0)|^{2p}$ by $\sup_{0\le s\le t}\mathbf{E}|X^\pi(s)|^{2p}$ to get the required inequality namely (0.13).

We also have

$$\begin{aligned}
&\int_0^t \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}\|X_s^\pi\|^2 du \\
&\le \int_0^t \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p-2}\int_{-1}^s |X^\pi(r)|^2 dr\, du \\
&\le \int_0^t\int_0^t \sup_{s\in[0,u]} \mathbf{E}|X^\pi(s)|^{2p-2}|X^\pi(r)|^2 dr\, du + \mathbf{E}\int_{-1}^0\int_{-1}^0 |X^\pi(s)|^{2p-2}|X^\pi(r)|^2 dr\, du \\
&\le \int_0^t\int_0^t \sup_{s\in[0,u]} \mathbf{E}\left(|X^\pi(s)|^{2p}+|X^\pi(r)|^{2p}\right) dr\, du + 2\mathbf{E}\int_{-1}^0\int_{-1}^0 |X^\pi(s)|^{2p} du du \\
&\le 2\|\theta\|^{2p} + 2\int_0^t\int_0^t \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p} dr\, du \\
&\le 2\|\theta\|^{2p} + 2t\int_0^t \sup_{0\le s\le u} \mathbf{E}|X^\pi(s)|^{2p} du. \qquad (0.14)
\end{aligned}$$

Now by combining inequalities (0.13) and (0.14) we get

$$\begin{aligned}
\sup_{0\le s\le t}\mathbf{E}|X^\pi(s)|^{2p} &\le \|V\|^{2p} + 6p(2p+1)(K+K^2)\int_0^t \sup_{0\le s\le u}(\mathbf{E}|X^\pi(s)|^{2p}+1)du \\
&\quad +6p(2p+1)(K+K^2)\|\theta\|^{2p} \\
&\quad +6p(2p+1)(K+K^2)t\int_0^t \sup_{0\le s\le u}\mathbf{E}|X^\pi(s)|^{2p}du \\
&\le 6p(2p+1)(K+K^2)\left(\|V\|^{2p}+\|\theta\|^{2p}+1\right) \\
&\quad +6p(2p+1)(K+K^2)t \\
&\quad +6p(2p+1)(K+K^2)(t+1)\int_0^t \sup_{0\le s\le u}\mathbf{E}|X^\pi(s)|^{2p}du \\
&\le 6p(2p+1)(K+K^2)(t+1)\left(\|V\|^{2p}+\|\theta\|^{2p}+1\right) \\
&\quad +6p(2p+1)(K+K^2)(t+1)\int_0^t \sup_{0\le s\le u}\mathbf{E}|X^\pi(s)|^{2p}du. \qquad (0.15)
\end{aligned}$$

We also have for each $t \in [0, a]$

$$\begin{aligned} \mathbf{E}\|X_t^\pi\|^{2p} &\leq \|\theta\|^{2p} + \int_0^t \mathbf{E}|X^\pi(r)|^{2p}dr \\ &\leq \|\theta\|^{2p} + \int_0^t \sup_{0\leq s\leq u} \mathbf{E}|X^\pi(s)|^{2p}du \end{aligned} \quad (0.16)$$

Now by using the same steps used to get (0.13)we find that

$$\sup_{0\leq s\leq u} \mathbf{E}\|X_s^\pi\|^{2p} \leq \|\theta\|^{2p} + \int_0^t \sup_{0\leq s\leq u} \mathbf{E}|X^\pi(s)|^{2p}du. \quad (0.17)$$

By combining inequalities (0.15) and (0.17) and using the inequality $(b+c)^{2p} \leq 2^{(2p-1)}(b^{2p} + c^{2p} \leq 2^{(2p-1)}(b+c)^{2p}$ we get

$$\begin{aligned} &\sup_{0\leq s\leq t} \mathbf{E}\,\|(X^\pi(s), X_s^\pi)\|^{2p} \\ &\leq 2^{2p-1} \sup_{0\leq s\leq t} \mathbf{E}\left(|X^\pi(s)|^{2p} + \|X_s^\pi)\|^{2p}\right) \\ &\leq \beta(t)\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) + \beta(t)\int_0^t \sup_{0\leq s\leq u} \mathbf{E}|X^\pi(s)|^{2p}du \\ &\leq \beta(t)\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) + \beta(t)\int_0^t \sup_{0\leq s\leq u} \mathbf{E}\,\|(X^\pi(s), X_s^\pi)\|^{2p}\,du, \end{aligned} \quad (0.18)$$

where $\beta(t) = [6p(2p+1)(K+K^2)(t+1)+1]\,2^{2p-1}$. Now applying Grönwall's inequality to (0.18), using $\beta(a)$ instead of $\beta(t)$ we get, for each $t \in [0, a]$,

$$\sup_{0\leq s\leq t} \mathbf{E}\,\|(X^\pi(s), X_s^\pi)\|^{2p} \leq C_1\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right)e^{C_1 t}, \quad (0.19)$$

where $C_1 = \beta(a)$ is a constant independent of δ. Thus we have

$$\sup_{0\leq s\leq a} \mathbf{E}\,\|(X^\pi(s), X_s^\pi)\|^{2p} \leq C\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) \quad (0.20)$$

where $C = \beta(a)e^{a\beta(a)}$ is a constant depending only on K, p and a. □

Proof of Proposition 3. Now by using estimate (0.20) and the definition of X^π and the properties of the conditional expectation we shall prove that X^π satisfies (0.6).

For each $k \in \{1, 2, ..., m\}$ we have, with $\Delta_k W = W(t_{k+1}) - W(t_k)$,

$$\begin{aligned}
&\mathbf{E}\left\{\left|\mathbf{E}\left(\frac{X^\pi(t_{k+1}) - X^\pi(t_k)}{\Delta_k} \mid \mathcal{F}_{t_k}\right) - f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^2\right\} \\
&= \mathbf{E}\Bigg(\left|\mathbf{E}\left(\frac{f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k - g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k W}{\Delta_k} \mid \mathcal{F}_{t_k}\right)\right. \\
&\qquad \left. - f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^2\Bigg) \\
&= \mathbf{E}\left(\left|\mathbf{E} g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{(W(t_{k+1}) - W(t_k))}{\Delta_k} \mid \mathcal{F}_{t_k}\right|^2\right) \\
&= \mathbf{E}\left(\left|g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\mathbf{E}\frac{(W(t_{k+1}) - W(t_k))}{\Delta_k} \mid \mathcal{F}_{t_k}\right|^2\right) \\
&= 0 = c_1(\delta) \quad \text{say.}
\end{aligned} \tag{0.21}$$

Notice that, to get the last step of inequality (0.21) we have used the fact that $\mathbf{E}\left\{(W(t_{k+1}) - W(t_k)) \mid \mathcal{F}_{t_k}\right\} = 0$ and $\mathbf{E}\left|g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^2$ is bounded by a constant because of (0.20). Thus X^π satisfies inequality (0.6).

Next we shall prove that X^π satisfies inequality (0.7):

$$\begin{aligned}
&\mathbf{E}\Bigg\{\left|\mathbf{E}\left(\frac{1}{\Delta_k}\left(X^\pi(t_{k+1}) - X^\pi t_k\right)\left(X^\pi(t_{k+1}) - X^\pi(t_k)\right)^T \mid \mathcal{F}_{t_k}\right)\right. \\
&\qquad \left. - g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right) g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)^T\right|^2\Bigg\} \\
&= \mathbf{E}\left\{\left|\frac{1}{\Delta_k}\mathbf{E}\left(\left\{f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k + g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k W\right\}^2 \,\Big|\, \mathcal{F}_{t_k}\right)\right.\right. \\
&\qquad \left.\left. - g^2(t_k, X^\pi(t_k), X^\pi_{t_k})\right|^2\right\} \\
&= \mathbf{E}\Big\{\Big| f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k \\
&\qquad + g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1}) - W(t_k)\right)^2 \mid \mathcal{F}_{t_k} \\
&\qquad + 2f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right) g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1}) - W(t_k)\right) \,\Big|\, \mathcal{F}_{t_k} \\
&\qquad - g^2(t_k, X^\pi(t_k), X^\pi_{t_k})\Big|^2\Big\} \\
&= \mathbf{E}\Big\{\Big| f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k \\
&\qquad + g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1}) - W(t_k)\right)^2 \,\Big|\, \mathcal{F}_{t_k} \\
&\qquad - g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Big|^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\Big\{\Big|f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k \\
&+g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\left\{\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1})-W(t_k)\right)^2 \mid \mathcal{F}_{t_k}-1\right\}\Big|^2\Big\} \\
&= \mathbf{E}\left\{\left|f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k\right|^2\right\} \quad (\text{as } \mathbf{E}\left[\left(W\left(t_{k+1}\right)-W\left(t_k\right)\right)^2 \mid \mathcal{F}_{t_k}\right]=\Delta_k) \\
&\le \mathbf{E}\left|f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^4 \Delta_k^2 \\
&\le \mathbf{E}\left\{K\left(|X^\pi(t_k)|+\|X^\pi_{t_k}\|+1\right)\right\}^4\delta^2 \quad (\text{by the linear growth condition on } f) \\
&\le 2^6K^4\left(\sup_{0\le s\le t_k}\mathbf{E}\,\|(X^\pi(s), X^\pi_s)\|^4+1\right)\delta^2 \\
&\le 2^7K^4C\left(\|V\|^4+\|\theta\|^4+1\right)\delta^2 \quad (\text{by estimate (0.20)}) \\
&= c_2(\delta) \quad \text{say.}
\end{aligned} \tag{0.22}
$$

Clearly, $c_2(\delta) \to 0$ as $\delta \to 0$. Let $C(\delta) = \max\{c_1(\delta), c_2(\delta)\}$. Hence $C(\delta) \to 0$ as $\delta \to 0^+$. Thus inequalities (0.5), (0.6) and (0.7) are satisfied with $C(\delta) = \max\{c_1(\delta), c_2(\delta)\} = 2^7K^4C\left(\|V\|^4+\|\theta\|^4+1\right)\delta^2$. Hence X^π is weakly consistent. □

0.3 Remarks:

(a) All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z : [0,a]\times\Omega \to \mathbf{R}$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities

$$
\begin{aligned}
|\mathbf{E}[Z(t)-Z(s)]|\mathcal{F}_s| &\le K(t-s) \quad \text{and} \\
\mathbf{E}\left(|Z(t)-Z(s)|^2|\mathcal{F}_s\right) &\le K(t-s) \quad \text{for } 0\le s\le t\le a.
\end{aligned}
$$

Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work.

(b) The Numerical Stability 0.1 and The Weak Consistency 0.2 can be extended to a processes $f', g' : [0,a]\times\mathbf{R}^n\times L^2(J,\mathbf{R}^n) \to L(\mathbf{R}^m,\mathbf{R}^n)$ $(m,n\in\mathbf{N})$ instead of the processes $f,g : [0,a]\times\mathbf{R}^n\times L^2(J,\mathbf{R}^n)\to\mathbf{R}^n$ $(n\in\mathbf{N})$, and instead of the Brownian motion W we use the process $Z : [0,a]\times\Omega\to\mathbf{R}^m$ which is a martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$, continuous on $[0,a]$, and has independent increments and satisfies for some constant K the inequalities

$$
|\mathbf{E}[Z(t)-Z(s)]|\mathcal{F}_s| \le K(t-s) \quad \text{and} \quad \mathbf{E}\left(|Z(t)-Z(s)|^2|\mathcal{F}_s\right)\le K(t-s)
$$

for $0\le s\le t\le a$.

(c) All the lemmas and theorems in this work hold for any delay interval $J' = [-r,0)$ $(r\ge 0)$.

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