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$$+3p(2p+1)(K + K^2)\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p-2}\|X_s\|^2 du. \quad (0.13)$$

Step 1: we know that inequality (0.12) is true with t replaced by s_0 in both sides of the inequality.

Step 2: Now step 1 is also true if we replace s_0 in the right hand side by t and leave s_0 in the left hand side as it is .

Step 3: Now replace $E|X(s_0)|^{2p}$ by $\sup_{0\le s\le t} E|X(s)|^{2p}$ to get the required inequality namely (0.13).

We also have

$$\begin{aligned}
\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p-2}\|X_s\|^2 du
&\le \int_0^t \sup_{0\le s\le u} E|X(s)|^{2p-2}\int_{-1}^{s}|X(r)|^2 dr\, du \\
&\le \int_0^t\int_0^t \sup_{s\in[0;u]} E|X(s)|^{2p-2}|X(r)|^2 dr\, du + E\int_{-1}^0\int_{-1}^0 |X(s)|^{2p-2}|X(r)|^2 dr\, du \\
&\le \int_0^t\int_0^t \sup_{s\in[0;u]} E\left(|X(s)|^{2p} + |X(r)|^{2p}\right) dr\, du + 2E\int_{-1}^0\int_{-1}^0 |X(s)|^{2p} du\, du \\
&\le 2\|\;\|^{2p} + 2\int_0^t\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p} dr\, du \\
&\le 2\|\;\|^{2p} + 2t\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p} du. \qquad (0.14)
\end{aligned}$$

Now by combining inequalities (0.13) and (0.14) we get

$$\begin{aligned}
\sup_{0\le s\le t} E|X(s)|^{2p} &\le \|V\|^{2p} + 6p(2p+1)(K + K^2)\int_0^t \sup_{0\le s\le u}(E|X(s)|^{2p}+1) du \\
&\quad +6p(2p+1)(K + K^2)\|\;\|^{2p} \\
&\quad +6p(2p+1)(K + K^2)t\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p} du \\
&\le 6p(2p+1)(K + K^2)\ \|V\|^{2p} + \|\;\|^{2p} + 1 \\
&\quad +6p(2p+1)(K + K^2)t \\
&\quad +6p(2p+1)(K + K^2)(t+1)\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p} du \\
&\le 6p(2p+1)(K + K^2)(t+1)\ \|V\|^{2p} + \|\;\|^{2p} + 1 \\
&\quad +6p(2p+1)(K + K^2)(t+1)\int_0^t \sup_{0\le s\le u} E|X(s)|^{2p} du. \qquad (0.15)
\end{aligned}$$

We also have for each $t \in [0, a]$

$$\begin{aligned}\mathbf{E}\|X^\pi_t\|^{2p} &\leq \|\theta\|^{2p} + \int_0^t \mathbf{E}|X^\pi(r)|^{2p} dr \\ &\leq \|\theta\|^{2p} + \int_0^t \sup_{0\leq s\leq u} \mathbf{E}|X^\pi(s)|^{2p} du \end{aligned} \tag{0.16}$$

Now by using the same steps used to get (0.13)we find that

$$\sup_{0\leq s\leq u} \mathbf{E}\|X^\pi_s\|^{2p} \leq \|\theta\|^{2p} + \int_0^t \sup_{0\leq s\leq u} \mathbf{E}|X^\pi(s)|^{2p} du. \tag{0.17}$$

By combining inequalities (0.15) and (0.17) and using the inequality $(b+c)^{2p} \leq 2^{(2p-1)}(b^{2p} + c^{2p} \leq 2^{(2p-1)}(b+c)^{2p}$ we get

$$\begin{aligned}&\sup_{0\leq s\leq t} \mathbf{E}\left\|(X^\pi(s), X^\pi_s)\right\|^{2p} \\ &\leq 2^{2p-1} \sup_{0\leq s\leq t} \mathbf{E}\left(|X^\pi(s)|^{2p} + \|X^\pi_s)\|^{2p}\right) \\ &\leq \beta(t)\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) + \beta(t) \int_0^t \sup_{0\leq s\leq u} \mathbf{E}|X^\pi(s)|^{2p} du \\ &\leq \beta(t)\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) + \beta(t) \int_0^t \sup_{0\leq s\leq u} \mathbf{E}\left\|(X^\pi(s), X^\pi_s)\right\|^{2p} du, \end{aligned} \tag{0.18}$$

where $\beta(t) = [6p(2p+1)(K+K^2)(t+1)+1]\, 2^{2p-1}$. Now applying Grönwall's inequality to (0.18), using $\beta(a)$ instead of $\beta(t)$ we get, for each $t \in [0, a]$,

$$\sup_{0\leq s\leq t} \mathbf{E}\left\|(X^\pi(s), X^\pi_s)\right\|^{2p} \leq C_1\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) e^{C_1 t}, \tag{0.19}$$

where $C_1 = \beta(a)$ is a constant independent of δ. Thus we have

$$\sup_{0\leq s\leq a} \mathbf{E}\left\|(X^\pi(s), X^\pi_s)\right\|^{2p} \leq C\left(\|V\|^{2p} + \|\theta\|^{2p} + 1\right) \tag{0.20}$$

where $C = \beta(a)e^{a\beta(a)}$ is a constant depending only on K, p and a. □

Proof of Proposition 3. Now by using estimate (0.20) and the definition of X^π and the properties of the conditional expectation we shall prove that X^π satisfies (0.6).

For each $k \in \{1, 2, ..., m\}$ we have, with $\Delta_k W = W(t_{k+1}) - W(t_k)$,

$$\begin{aligned}
&\mathbf{E}\left\{\left|\mathbf{E}\left(\frac{X^\pi(t_{k+1}) - X^\pi(t_k)}{\Delta_k} \mid \mathfrak{F}_{t_k}\right) - f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^2\right\} \\
&= \mathbf{E}\Bigg(\Bigg|\mathbf{E}\Bigg(\frac{f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k - g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k W}{\Delta_k} \Bigm| \mathfrak{F}_{t_k}\Bigg) \\
&\quad - f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Bigg|^2\Bigg) \\
&= \mathbf{E}\left(\left|\mathbf{E}g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{(W(t_{k+1}) - W(t_k))}{\Delta_k} \mid \mathfrak{F}_{t_k}\right|^2\right) \\
&= \mathbf{E}\left(\left|g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\mathbf{E}\frac{(W(t_{k+1}) - W(t_k))}{\Delta_k} \mid \mathfrak{F}_{t_k}\right|^2\right) \\
&= 0 = c_1(\delta) \quad \text{say.}
\end{aligned} \tag{0.21}$$

Notice that, to get the last step of inequality (0.21) we have used the fact that $\mathbf{E}\left\{(W(t_{k+1}) - W(t_k)) \mid \mathfrak{F}_{t_k}\right\} = 0$ and $\mathbf{E}\left|g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^2$ is bounded by a constant because of (0.20). Thus X^π satisfies inequality (0.6).

Next we shall prove that X^π satisfies inequality (0.7):

$$\begin{aligned}
&\mathbf{E}\Bigg\{\Bigg|\mathbf{E}\left(\frac{1}{\Delta_k}\left(X^\pi(t_{k+1}) - X^\pi t_k\right)\left(X^\pi(t_{k+1}) - X^\pi(t_k)\right)^T \mid \mathfrak{F}_{t_k}\right) \\
&\quad - g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right) g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)^T\Bigg|^2\Bigg\} \\
&= \mathbf{E}\left\{\left|\frac{1}{\Delta_k}\mathbf{E}\left(\left\{f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k + g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k W\right\}^2 \Bigm| \mathfrak{F}_{t_k}\right)\right.\right. \\
&\quad \left.\left. -g^2(t_k, X^\pi(t_k), X^\pi_{t_k})\right|^2\right\} \\
&= \mathbf{E}\Big\{\Big|f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k \\
&\quad +g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1}) - W(t_k)\right)^2 \Bigm| \mathfrak{F}_{t_k} \\
&\quad +2f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right) g\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1}) - W(t_k)\right) \Bigm| \mathfrak{F}_{t_k} \\
&\quad -g^2(t_k, X^\pi(t_k), X^\pi_{t_k})\Big|^2\Big\} \\
&= \mathbf{E}\Big\{\Big|f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k \\
&\quad +g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1}) - W(t_k)\right)^2 \Bigm| \mathfrak{F}_{t_k} \\
&\quad -g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Big|^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left\{\left|f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k\right.\right.\\
&\left.\left.+g^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\left\{\frac{1}{\Delta_k}\mathbf{E}\left(W(t_{k+1})-W(t_k)\right)^2 \mid \mathcal{F}_{t_k}-1\right\}\right|^2\right\}\\
&= \mathbf{E}\left\{\left|f^2\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\Delta_k\right|^2\right\} \quad \left(\text{as } \mathbf{E}\left[\left(W\left(t_{k+1}\right)-W\left(t_k\right)\right)^2 \mid \mathcal{F}_{t_k}\right]=\Delta_k\right)\\
&\leq \mathbf{E}\left|f\left(t_k, X^\pi(t_k), X^\pi_{t_k}\right)\right|^4\Delta_k^2\\
&\leq \mathbf{E}\left\{K\left(|X^\pi(t_k)|+\|X^\pi_{t_k}\|+1\right)\right\}^4\delta^2 \quad (\text{by the linear growth condition on } f)\\
&\leq 2^6K^4\left(\sup_{0\leq s\leq t_k}\mathbf{E}\left\|(X^\pi(s), X^\pi_s)\right\|^4+1\right)\delta^2\\
&\leq 2^7K^4C\left(\|V\|^4+\|\theta\|^4+1\right)\delta^2 \quad (\text{by estimate } (0.20))\\
&= c_2(\delta) \quad \text{say.}
\end{aligned} \tag{0.22}
$$

Clearly, $c_2(\delta) \to 0$ as $\delta \to 0$. Let $C(\delta) = \max\{c_1(\delta), c_2(\delta)\}$. Hence $C(\delta) \to 0$ as $\delta \to 0^+$. Thus inequalities (0.5), (0.6) and (0.7) are satisfied with $C(\delta) = \max\{c_1(\delta), c_2(\delta)\} = 2^7K^4C\left(\|V\|^4+\|\theta\|^4+1\right)\delta^2$. Hence X^π is weakly consistent. □

0.3 Remarks:

(a) All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z : [0,a]\times\Omega \to \mathbf{R}$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities

$$
\begin{aligned}
|\mathbf{E}[Z(t)-Z(s)]|\mathcal{F}_s| &\leq K(t-s) \quad \text{and}\\
\mathbf{E}\left(|Z(t)-Z(s)|^2|\mathcal{F}_s\right) &\leq K(t-s) \quad \text{for } 0\leq s\leq t\leq a.
\end{aligned}
$$

Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work.

(b) The Numerical Stability 0.1 and The Weak Consistency 0.2 can be extended to a processes $f', g' : [0,a]\times\mathbf{R}^n\times L^2(J,\mathbf{R}^n)\to L(\mathbf{R}^m,\mathbf{R}^n)$ $(m,n\in\mathbf{N})$ instead of the processes $f, g : [0,a]\times\mathbf{R}^n\times L^2(J,\mathbf{R}^n)\to\mathbf{R}^n$ $(n\in\mathbf{N})$, and instead of the Brownian motion W we use the process $Z : [0,a]\times\Omega\to\mathbf{R}^m$ which is a martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$, continuous on $[0,a]$, and has independent increments and satisfies for some constant K the inequalities

$$
|\mathbf{E}[Z(t)-Z(s)]|\mathcal{F}_s| \leq K(t-s) \quad \text{and} \quad \mathbf{E}\left(|Z(t)-Z(s)|^2|\mathcal{F}_s\right)\leq K(t-s)
$$

for $0\leq s\leq t\leq a$.

(c) All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0)$ $(r\geq 0)$.

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