

# Dynamical Behavior in a Discrete Three Species Prey-Predator System

A.George Maria Selvam<sup>1</sup>, R. Dhineshababu<sup>2</sup> and V.Sathish<sup>3</sup>

<sup>1,3</sup>Sacred Heart College, Tirupattur – 635 601, S.India.

<sup>2</sup>DMI College of Engineering, Chennai - 600 123, S.India.

## ABSTRACT

This paper investigates the dynamical behavior of a discrete prey-predator system with three species. Stability analysis is performed and analytical results are illustrated with numerical simulations. Time series plots and phase portraits are obtained for different sets of parameter values. Bifurcation diagrams are provided for selected range of growth parameter.

**Keywords:** Discrete prey - predator system, difference equations, equilibrium points, stability.

## 1. INTRODUCTION

Simple models of food webs may exhibit very complex dynamics. The Lotka – Volterra equations [4] (ODE) successfully described an ecological predator – prey model and the oscillations in their populations. In recent decades, many researchers [1, 3, 6, 7, 9] have focused on the ecological models with three and more species to understand complex dynamical behaviors of ecological systems in the real world. They have demonstrated very complex dynamic phenomena of those models, including cycles, periodic doubling and chaos [1, 2]. The dynamical behavior and stability analysis of nonlinear discrete prey-predator and host parasite model has been studied. The discrete time models which are usually described by difference equations can produce much richer patterns [2, 4, 5, 8]. Discrete time models are ideally suited to describe the population dynamics of species, which are characterized by discrete generations.

## 2. MATHEMATICAL MODEL

In this paper, we consider the discrete-time prey-predator system describing the interactions among three species by the following system of difference equations:

$$\begin{aligned} x(t+1) &= rx(t)[1-x(t)]-bx(t)y(t) \\ y(t+1) &= (1-c)y(t)+dx(t)y(t)-ey(t)z(t) \\ z(t+1) &= (1-f)z(t)+gy(t)z(t) \end{aligned} \quad (1)$$

where  $x(t)$ ,  $y(t)$  and  $z(t)$  are functions of time representing population densities of the prey and the mid-predator and the top predator, respectively, and all parameters are positive constants. The parameter  $r$  is the

intrinsic growth rates of the prey population,  $b$  is the per capita rate of predation of the mid-predator,  $c$  denotes the death rate of the mid-predator,  $d$  is the rate of conversion of a consumed prey to a predator,  $e$  is the per-capita rate of predation of the top predator,  $f$  denotes the death rate of the top predator,  $g$  is the rate of conversion of a consumed prey to a predator.

## 3. EXISTENCE OF EQUILIBRIUM

The equilibrium points of (1) are the solution of the equations

$$\begin{aligned} x &= rx[1-x]-bxy \\ y &= (1-c)y+dx-yz \\ z &= (1-f)z+gyz. \end{aligned}$$

The equilibrium points are

$$\begin{aligned} E_0 &= (0, 0, 0), \quad E_1 = \left(1 - \frac{1}{r}, 0, 0\right), \\ E_2 &= \left(\frac{c}{d}, \frac{r(d-c)}{bd} - \frac{1}{b}, 0\right) \text{ and } E_3 = (x^*, y^*, z^*). \end{aligned}$$

where,

$$\begin{aligned} x^* &= 1 - \frac{1}{r} \left(1 + \frac{bf}{g}\right), \quad y^* = \frac{f}{g}, \text{ and} \\ z^* &= \frac{d-c}{e} - \frac{d}{re} \left(1 + \frac{bf}{g}\right). \end{aligned}$$

Interior equilibrium point  $E_3$

corresponds to the coexistence of all species.

## 4. DYNAMICAL BEHAVIOR OF THE MODEL

In this section, we study the local behavior of the system (1) about each equilibrium points. The stability of the system (1) is carried out by computing the Jacobian matrix corresponding to each equilibrium point. The Jacobian Matrix  $J$  for the system (1) is

$$J(x, y, z) = \begin{bmatrix} r-2rx-by & -bx & 0 \\ dy & 1-c+dx-ez & -ey \\ 0 & gz & 1-f+gy \end{bmatrix}. \quad (2)$$

The determinant of the Jacobian  $J(x, y)$  is

$$Det = (1-f+gy)[r(1-2x)(1-c+dx)-by(1-c)].$$

Hence the system (1) is dissipative dynamical system when

$$|(1-f+gy)[r(1-2x)(1-c+dx)-by(1-c)]| < 1.$$

**Theorem 1:** The equilibrium point  $E_0$  is locally asymptotically stable if  $r < 1$ ,  $0 < c < 2$  and  $0 < f < 2$ , otherwise unstable equilibrium point.

**Proof:** In order to prove this result, we determine the eigenvalues of Jacobian matrix  $J$  at  $E_0$ . The Jacobian matrix evaluated at the equilibrium point  $E_0$  has the form

$$J(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & 1-c & 0 \\ 0 & 0 & 1-f \end{pmatrix}.$$

Hence the eigenvalues of the matrix  $J(E_0)$  are  $\lambda_1 = r$ ,  $\lambda_2 = 1-c$  and  $\lambda_3 = 1-f$ . Thus  $E_0$  is stable when  $r < 1$ ,  $0 < c < 2$  and  $0 < f < 2$ . Otherwise  $E_0$  is unstable equilibrium point.

**Theorem 2:** The equilibrium point  $E_1$  is locally asymptotically stable if  $1 < r < 3$ ,  $r < \frac{d}{d-c}$  and  $0 < f < 2$ , otherwise unstable equilibrium point.

**Proof:** The Jacobian matrix  $J$  for the system evaluated at the equilibrium point  $E_1$  is given by

$$J(E_1) = \begin{pmatrix} 2-r & b\left(\frac{1}{r}-1\right) & 0 \\ 0 & 1-c+d\left(1-\frac{1}{r}\right) & 0 \\ 0 & 0 & 1-f \end{pmatrix}.$$

Hence the eigenvalues of the matrix  $J(E_1)$  are

$$\lambda_1 = 2-r, \lambda_2 = 1-c+d\left(1-\frac{1}{r}\right) \text{ and } \lambda_3 = 1-f.$$

Hence  $E_1$  is locally asymptotically stable when  $1 < r < 3$ ,  $r < \frac{d}{d-c}$  and  $0 < f < 2$ , and unstable when  $r > 3$ ,  $r > \frac{d}{d-c}$  and  $f > 2$ .

**Theorem 3:** The equilibrium point  $E_2$  is locally asymptotically stable if  $\frac{d}{d-c} < r < \frac{d(bf+g)}{g(d-c)}$ , otherwise unstable equilibrium point.

**Proof:** The Jacobian matrix evaluated at  $E_2$  is given by

$$J(E_2) = \begin{pmatrix} 1-\frac{rc}{d} & -\frac{bc}{d} & 0 \\ \frac{r(d-c)-d}{b} & 1 & \frac{e}{b}-\frac{re(d-c)}{bd} \\ 0 & 0 & 1-f+\frac{gr(d-c)}{bd}-\frac{g}{b} \end{pmatrix}.$$

Hence the eigenvalues of the matrix  $J(E_2)$  are

$$\lambda_1 = 1-f-\frac{g}{b}+\frac{gr}{bd}(d-c) \text{ and}$$

$$\lambda_{2,3} = 1-\frac{cr}{2d} \pm \frac{1}{2d} \sqrt{4cdr(c-d)+c^2r^2+4cd^2}.$$

Hence  $E_2$  is locally asymptotically stable when  $\frac{d}{d-c} < r < \frac{d(bf+g)}{g(d-c)}$ , and unstable when  $\frac{d}{d-c} > r$  or  $r > \frac{d(bf+g)}{g(d-c)}$ .

## 5. LOCAL STABILITY AND DYNAMICAL BEHAVIOR AROUND INTERIOR FIXED POINT $E_3$

We now investigate the local stability and bifurcation of interior fixed point  $E_3$ . The Jacobian matrix  $J$  at  $E_3$  has of the form

$$J(E_3) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \quad (3)$$

$$\text{where, } a_{11} = 2-r+\frac{bf}{g}, \quad a_{12} = \frac{b}{r}\left(1+\frac{bf}{g}\right)-b,$$

$$a_{13} = 0, \quad a_{21} = \frac{df}{g}, \quad a_{22} = 1, \quad a_{23} = -\frac{fe}{g},$$

$$a_{31} = 0, \quad a_{32} = \frac{g(d-c)}{e}-\frac{d(g+bf)}{re}, \quad a_{33} = 1.$$

Its characteristic equation is

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (4)$$

with

$$A = -(a_{11} + a_{12} + a_{13}),$$

$$B = a_{11}a_{33} + a_{22}a_{33} + a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32},$$

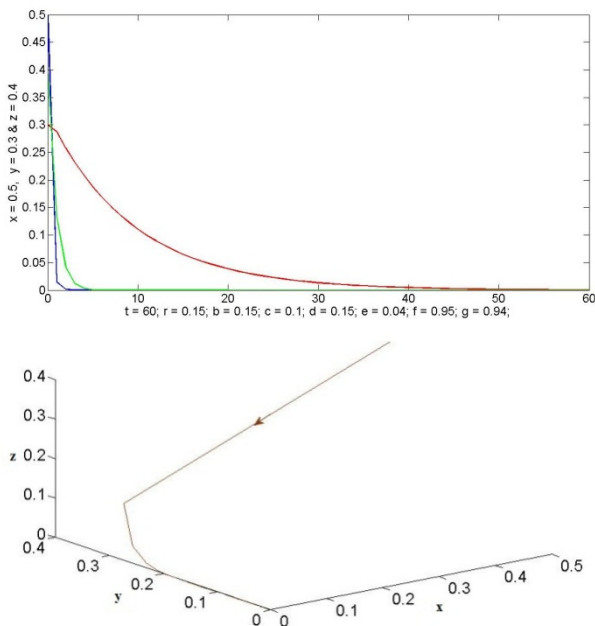
$$C = (a_{12}a_{21} - a_{11}a_{22})a_{33} + a_{11}a_{23}a_{32}.$$

By the Routh-Hurwitz criterion,  $E_3 = (x^*, y^*, z^*)$  is locally asymptotically stable if and only if  $A, C$ , and  $AB - C$  are positive.

### 6. NUMERICAL SIMULATIONS

In this section, we present the time plots, phase portraits and bifurcation diagrams to illustrate the theoretical analysis and show the interesting complex dynamical behaviors of the system (1).

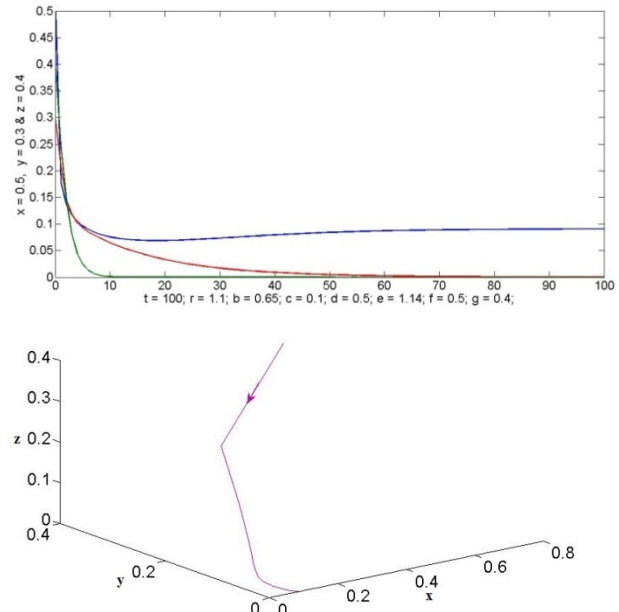
**Example1:** We shall consider  $r = 0.15, b = 0.15, c = 0.1, d = 0.15, e = 0.04, f = 0.95$  and  $g = 0.94$ . At equilibrium point  $E_0$ , the eigenvalues are  $\lambda_1 = 0.15, \lambda_2 = 0.9$  and  $\lambda_3 = 0.05$  so that  $|\lambda_{1,2,3}| < 1$ . Hence the trivial equilibrium point is stable (see Figure-1).



**Figure 1:** Time Series Plot and Phase Portrait at  $E_0$ .

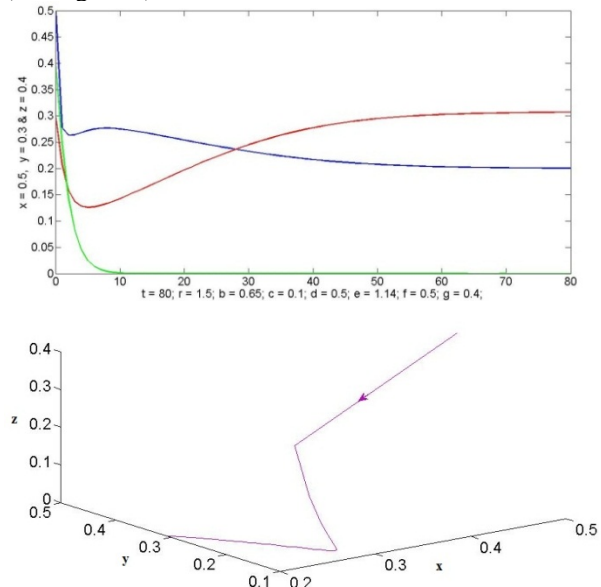
**Example2:** We shall consider the parameter values  $r = 1.1, b = 0.65, c = 0.1, d = 0.5, e = 1.14, f = 0.5$  and  $g = 0.4$ . The equilibrium point  $E_1 = (0.091, 0, 0)$  and the eigenvalues are  $\lambda_1 = 0.9, \lambda_2 = 0.945$  and  $\lambda_3 = 0.5$

so that  $|\lambda_{1,2,3}| < 1$ . Hence system (1) is stable (see Figure-2).



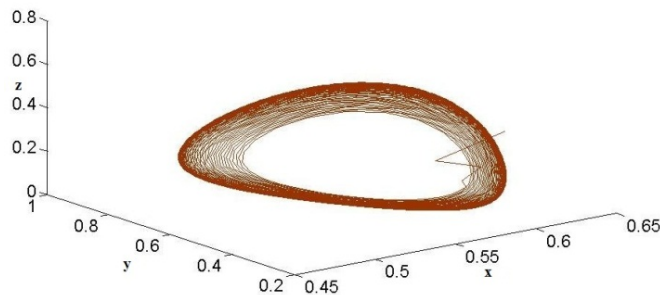
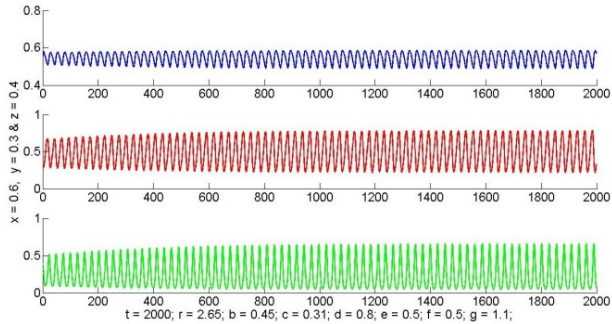
**Figure 2:** Time Series Plot and Phase Portrait at  $E_1$ .

**Example3:** We shall consider the parameter values  $r = 1.5, b = 0.65, c = 0.1, d = 0.5, e = 1.14, f = 0.5$  and  $g = 0.4$ . The equilibrium point  $E_2 = (0.2, 0.308, 0)$  and the eigenvalues are  $\lambda_1 = 0.623, \lambda_2 = 0.9$  and  $\lambda_3 = 0.8$  so that  $|\lambda_{1,2,3}| < 1$ . Thus system (1) is stable (see Figure-3).



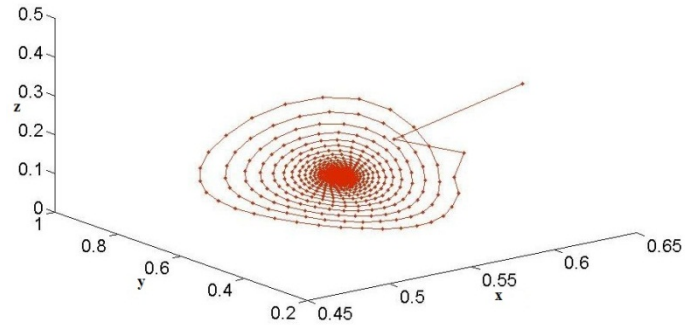
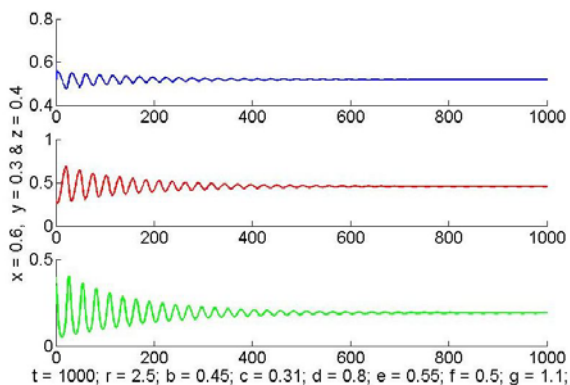
**Figure 3:** Time Series Plot and Phase Portrait at  $E_2$ .

**Example4:** When  $r = 2.65$ ,  $b = 0.45$ ,  $c = 0.31$ ,  $d = 0.8$ ,  $e = 0.5$ ,  $f = 0.5$  and  $g = 1.1$ . The eigenvalues are  $\lambda_1 = -0.3828$  and  $\lambda_{2,3} = 0.9686 \pm i0.2549$  so that  $|\lambda_1| < 1$  and  $|\lambda_{2,3}| = 1.0016 > 1$ . The system (1) is unstable (see Figure-4).



**Figure 4:** Time Series Plot and Phase Portrait at  $E_3$ .

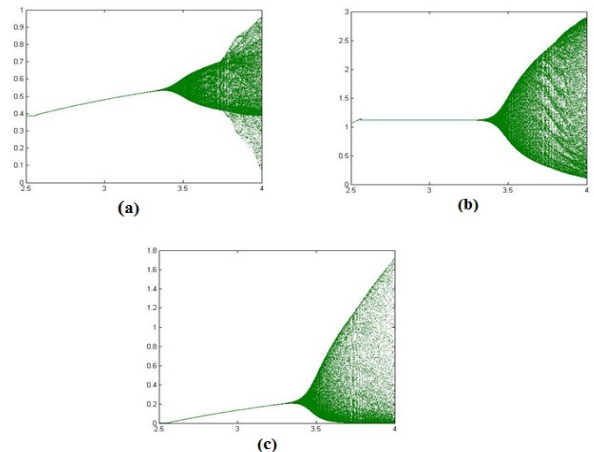
While with  $r = 2.5$ ,  $e = 0.55$  and keeping all other parameters same, we obtain  $E_3 = (0.518, 0.454, 0.191)$  and the eigenvalues are  $\lambda_1 = -0.2288$  and  $\lambda_{2,3} = 0.9667 \pm i0.2329$  so that  $|\lambda_1| < 1$  and  $|\lambda_{2,3}| = 0.9943 < 1$ . We observe the system (1) is stable (see Figure-5).



**Figure 5:** Time Series Plot and Phase Portrait at  $E_3$ .

## 7. BIFURCATION ANALYSIS VIA NUMERICAL SIMULATIONS

Bifurcation is a change of the dynamical behaviors of the system as its parameters pass through a *bifurcation (critical) value*. Bifurcation usually occurs when the stability of an equilibrium changes. In this section, we focus on exploring the possibility of chaotic behavior for the prey and the mid-predator and the top predator respectively.



**Figure 6:** Bifurcation diagrams for (a) prey population, (b) mid-predator population and (c) top predator.

Figure (6) presents the bifurcation diagram for prey and the mid-predator and the top predator densities of the system (1) with initial conditions  $x = 0.6$ ,  $y = 0.3$  and  $z = 0.4$  as above and we consider the parameters values  $b = 0.5$ ,  $c = 0.31$ ,  $d = 0.8$ ,  $e = 0.55$ ,  $f = 0.95$ ,  $g = 0.85$  and  $r = 2.5:0.001:4$ . The bifurcation diagrams imply the existence of chaos. Also these results reveal far richer dynamics of the discrete-time models.

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