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Approximate Analytical Solutions of the Fractional Nonlinear Dispersive Equations Using Homotopy Perturbation Sumudu Transform Method (HPSTM)

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Abstract

In this paper, we propose a combined form of the Sumudu transform method with the Homotopy perturbation method to solve nonlinear equations. This method is called the Homotopy perturbation Sumudu transform method (HPSTM). The nonlinear terms can be easily handled by the use of He's polynomials. The technique finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The results obtained by the two methods are in the agreement. Approximate analytical solution of nonlinear dispersive equations is calculated in the form of a convergence power series with easily computable components. Some illustrative examples are presented to explain the efficiency and simplicity of the proposed method.

Keywords: Sumudu transform, Homotopy perturbation method, Homotopy Perturbation Sumudu transform method, Nonlinear dispersive equations, He's polynomials.

1 Introduction

In the last several years with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations. Searching of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic that's need new methods for finding the exact and approximate solutions. Most of new nonlinear equations do not have a precise analytic solution; so, numerical methods have largely been used to handle these equations. There are also analytic techniques for nonlinear equations. Some of the classic analytic methods are Lyapunov's artificial small parameter method [1], Expansion method [2], Perturbation techniques [3-5] and Hirota bilinear attention to study the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM) [6], He's semi-inverse method [7], the Tanh method, the Homotopy perturbation Method (HPM), the Sinh–cosh method, the differential transform method and The Adomian decomposition method, the Variational iteration method, the Weighted finite difference techniques and the Lap lace decomposition method Have been used to handle advection equations [8-14]. Most of these methods have their inbuilt deficiencies like the calculation of A domian's polynomials, The Lagrange multiplier, divergent results and huge computational work. He [15-23] developed the Homotopy perturbation method (HPM) by merging the Standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The Lap lace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [24] and the Lap lace decomposition algorithm [25-29]. Furthermore, the Homotopy perturbation method is also combined with the well-known Lap lace transformation method [30, 31] and the variational iteration method [32] to produce a

1

highly effective technique for handling many nonlinear problems. There are numerous integral transforms such as the Lap lace, Sumudu, Fourier, Mellin, and Hankel to solve PDEs. Of these, the Laplace transformation and Sumudu transformation are the most widely used. The Sumudu transformation method is one of the most important transform methods introduced in the early 1990s by Gamage K. Watugala. It is a powerful tool for solving many kinds of FPDEs in various fields of science and engineering [33, 37], also various methods are combined with the Sumudu transformation method such as the Homotopy analysis Sumudu transform Method (HASTM) [38] which is a combination of the Homotopy analysis method and the Sumudu transformation method. Another example is the Sumudu decomposition method (ADSM) [39], which is a combination of the Sumudu transform method and the A domian decomposition method, and etc.

In the present paper, we propose a new method called Homotopy perturbation Sumudu transform method (HPSTM) for solving the nonlinear equations. It is worth mentioning that the proposed method is an elegant combination Of the Sumudu transformation, the Homotopy perturbation method and He’s Polynomials and is mainly due to Ghorbani [40, 41]. The use of He’s polynomials in the nonlinear term was first introduced by Ghorbani [40, 41]. The Proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact and numerical solutions for nonlinear dispersive equations. This paper considers the effectiveness of the Homotopy Perturbation Sumudu transform method (HPSTM) in solving nonlinear dispersive equations.

2 Basic Definitions of Fractional Calculus

In this section, we present the basic definitions and properties of fractional calculus theory which will be used in this paper.

2.1 Definition:

The Riemann-Liouville fractional integral operator of order $\alpha > 0$,of a function $f(t) \in c_p$,and $p \geq -1$,is defined as [42]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, (\alpha > 0),$$

$$I^0 f(t) = f(t).$$

For the Riemann- Liouville fractional integral we have

$$I^{\alpha}.t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$$

2.2 Definition

The Mittag-Leffler function $E_{\alpha}(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [43]:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

3 Definitions of Sumudu transform

3.1 *Definition*

In early 90's [44] Watauga introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions:

$$A = \left\{ f(t) : \exists M \tau_1, \tau_2 > 0, \left| \ f(t) \ \right| < M e^{|t|/\tau_j}, if \ t \in (-1)^j \times [0, \infty) \right\}$$

by the following formula:

$$G(u) = S[f(t);u] = \int_0^\infty f(ut)e^{-t}dt, u \in (-\tau_{1,}, \tau_2).$$

3. 2 Definition

The Sumudu transform of the Caputo fractional derivative is defined as follows [45]:

$$S[D_t^\alpha f(t)] = u^{-\alpha}S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0+), (m-1 < \alpha \le m) \text{ where } G(u) = S[f(t)].$$

3. 3 Some properties of the Sumudu transform

There are several properties of Sumudu transform such that;

3. 3. 1 Linearity property

$$S[af(t) + bg(t)] = aS[f(t)] + bS[g(t)]$$

3. 3. 2 *The integral formula*

$$S[\int_0^t f(\tau)d\tau] = uS[f(t)].$$

4 Dispersive equations

The subject nonlinear partial differential equations, in particular, nonlinear evolution equations of dispersive type are a very active field of research and study in recent times. A partial differential equation (PDE) is called dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. As an example consider $iU_t + U_{xx} = 0$. [46]**.** If we try a simple wave of the form $U(x,t) = Ae^{i(kx-wt)}$ we see that it satisfies the equation if and only if $w = k^2$. This is called the dispersive relation and shows that the frequency is a real valued function of the wave number. If we denote the Phase velocity by $v = W/K$. we can write the solution as $U(x,t) = Ae^{ik(x-v(k)t)}$ and notice that the wave travels with velocity k. Thus the wave propagates in such a way that large wave numbers travel faster than smaller ones. If we add nonlinear effects and study $iU_t + U_{xx} = F(u)$, we will see that even the existence of solutions over small times requires delicate techniques. Finally, we can say that nonlinear dispersive equations are model equations for propagation of waves in nonlinear dispersive media that appear in universal models such as water waves, optics, nonlinear plasma physics, and energy transfer in molecular systems, Bose-Einstein condensation [47].

5 The basic idea of Homotopy perturbation method

To illustrate the basic idea of this method, we consider the following nonlinear differential equations:

$$A(U) - f(r) = 0, r \in \Omega \quad (1)$$

with boundary condition:

$$B(U, \frac{\partial U}{\partial n}) = 0, r \in \Gamma, \quad (2)$$

where A is a general differential operator, B a boundary operator, f(r) is a known analytic function, Γ is the boundary of the domain Ω. In general the operator A can be dividing into two parts L and N where L is linear, N is nonlinear. Eq (1) can be rewritten as follows:

$$L(U)+N(U)-f(r)=0 \quad (3)$$

By the homotopy technique [48, 49] we construct the homotopy $v(r,p):\Omega\times[0,1]\rightarrow R$ which satisfies :

$$H(v,p)=(1-p)[L(v)-L(U_0)]+p[A(U)-f(r)]=0,\ p\in[0,1], r\in\Omega$$

(4)

Or

$$H(v,p)=[L(v)-L(U_0)]+pL(U_0)+p[A(U)-f(r)]=0,\ p\in[0,1] \quad (5)$$

Where p is an embedding parameter, and U_0 is an initial approximation of (1) which satisfies the boundary conditions. From (4) and (5) we have:

$$H(v,0)=[L(v)-L(U_0)]=0, \quad (6)$$

$$H(v,1)=[A(v)-f(r)]=0, \quad (7)$$

The changing in the process of p from zero from U_0 to $U(r)$. In topology this is called deformation and $[A(v)-f(r)]$, $[L(v)-L(U_0)]$ are called homotopic. Now, assume that the solution of (4), (5) can be expressed as

:

(8) $v=v_0+pv_1+p^2v_2+...$

Setting p=1 result in the approximate solution of (1):

$$U=\lim_{p\to 1} v=v_0+v_1+v_2+.... \quad (9)$$

6 Homotopy Perturbation Sumudu Transform Method

To illustrate the basic idea of this method, we consider the following nonlinear fractional differential equation:

$$D_t^{\alpha}U(x,t)+L(U(x,t))+N(U(x,t))=q(x,t),\ t>0, 0<\alpha<1 \quad (10)$$

Subject to initial condition :

$U(x,0)=f(x)$

where $D_t^{\alpha}=\partial^{\alpha}/\partial^{\alpha}$ is the fractional Caputo derivative of the function $U(x,t)$, L is the linear differential operator, N is the nonlinear differential operator, and $q(x,t)$ is the source term. Now, applying the Sumudu transform on both sides of (10) we have:

$$S[D_t^{\alpha}U(x,t)]+S[L(U(x,t))]+S[N(U(x,t))]=S[q(x,t)]. \quad (11)$$

Using the differential property of Sumudu transform, we have

$$S[U(x,t)]=f(x)-u^{\alpha}S[L(U(x,t))+N(U(x,t))]+u^{\alpha}S[q(x,t)]. \quad (12)$$

Operating with Sumudu inverse on both sides of (12)

$$U(x,t)=Q(x,t)-S^{-1}[u^{\alpha}S[L(U(x,t))+N(U(x,t))]]. \tag{13}$$

where $Q(x,t)$ represents the term arising from the source term and the prescribed initial conditions.
Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t)=\sum_{n=0}^{\infty}p^{n}U_{n}(x,t), \tag{14}$$

where the homotopy parameter p is considered as a small parameter $p\in[0,1]$.
We can decompose the nonlinear term as:

$$NU(x,t)=\sum_{n=0}^{\infty}p^{n}H_{n}(U), \tag{15}$$

where H_n are He's polynomials of $U_0(x,t),U_1(x,t),U_2(x,t),....,U_n(x,t)$ [50–52], and it can be calculated by the following formula:

$$H_n(U_0(x,t),U_1(x,t),U_2(x,t),....,U_n(x,t))=\frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty}p^{i}U_i)]_{p=0},n=0,1,2,... \tag{16}$$

By substituting (15) and (16) and using HPM [15] we get:

$$\sum_{n=0}^{\infty}p^{n}U_{n}(x,t)=Q(x,t)-p(S^{-1}[u^{\alpha}S[L(\sum_{n=0}^{\infty}p^{n}U_{n}(x,t))+(\sum_{n=0}^{\infty}p^{n}H_{n}(U(x,t)))]]). \tag{17}$$

This is coupling of Sumudu transform and homotopy perturbation method using He's polynomials. By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^{0}:U_0(x,t)=Q(x,t). \tag{18}$$

$$p^{1}:U_1(x,t)=-(S^{-1}[u^{\alpha}S[L(U_0(x,t))+(H_0(U(x,t)))]]), \tag{19}$$

$$p^{2}:U_2(x,t)=-(S^{-1}[u^{\alpha}S[L(U_1(x,t))+(H_1(U(x,t)))]]), \tag{20}$$

$$p^{3}:U_3(x,t)=-(S^{-1}[u^{\alpha}S[L(U_2(x,t))+(H_2(U(x,t)))]]), \tag{21}$$

proceeding in the same manner, the rest of the components $U_n(x,t)$ can be completely obtained, and the series solution is thus entirely determined. Finally we approximate the solution $U(x,t)$ by truncated series [53]

$$U(x,t)=\lim_{N\to\infty}\sum_{n=0}^{N}U_{n}(x,t). \tag{22}$$

These series solutions generally converge very rapidly.

7 Applications

In this section we apply this method for solving some fractional nonlinear dispersive equations.

Application: 1. Fisher's equation

The Fisher equation describes the process of interaction between diffusion and reaction. This equation is encountered in chemical kinetics and population dynamics [54] which includes problems such as nonlinear evolution of a population in one dimensional habitual, neutron population in a nuclear reaction.
Let us consider the nonlinear time-fractional Fisher's equation as follows:

$$D_t^{\alpha}U(x,t)=U_{xx}(x,t)+6U(x,t)(1-U(x,t)),0<\alpha<1,t>0. \tag{23}$$

Subjected to the initial condition:

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$$U(x,0) = \frac{1}{(1+e^x)^2}$$

We can solve Eq (23) by HPSTM by applying the Sumudu transform on both sides of (23), we obtain:

$$S[D_t^\alpha U(x,t)] = S[U_{xx}(x,t) + (6(U(x,t)((1-U(x,t)))] . \quad (24)$$

Using the property of the Sumudu transform, we have

$$S[U(x,t)] = U(x,0) + u^\alpha S[U_{xx}(x,t) + (6(U(x,t)) - (6(U^2(x,t)))] . \quad (25)$$

Now applying the Sumudu inverse on both sides of (26) we obtain:

$$U(x,t) = U(x,0) + S^{-1}[u^\alpha S[U_{xx}(x,t) + (6(U(x,t)) - (6(U^2(x,t)))]]. \quad (26)$$

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) . \quad (27)$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$).

We can decompose the nonlinear term as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U) , \quad (28)$$

where H_n are He's polynomials of $U_0(x,t), U_1(x,t), U_2(x,t), ..., U_n(x,t)$ and it can be calculated by the following formula:

$$H_n(U_0(x,t), U_1(x,t), U_2(x,t), ..., U_n(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty} p^i U_i)]_{p=0}, n = 0,1,2,... \quad (29)$$

By substituting (28) and (29) and using HPM we get:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = U(x,0) + pS^{-1}[u^\alpha S[U_{xx}(x,t)) + ((6(U(x,t)) - (6(H_n(U(x,t))))] \quad (30)$$

By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : U_0(x,t) = U(x,0) = \frac{1}{(1+e^x)^2} , \quad (31)$$

$$p^1 : U_1(x,t) = S^{-1}[u^\alpha S[U_{0xx}(x,t) + 6(U_0) - 6(U_o^2)]] \quad (32)$$

After simplification with using mathematica package we have:

$$U_1(x,t) = \frac{10e^x t^\alpha}{(1+e^x)^3 \Gamma(1+\alpha)}, \quad (33)$$

$$U_2(x,t) = \frac{50e^x(-1+2e^x)t^{2\alpha}}{(1+e^x)^4 \Gamma(1+2\alpha)}, \quad (34)$$

$$U_3(x,t) = \frac{50e^x t^{3\alpha}((5-6e^x-15e^{2x}+20e^{3x})\Gamma(1+\alpha)^2 - 12e^x\Gamma(1+2\alpha))}{(1+e^x)^6 \Gamma(1+\alpha)^2 \Gamma(1+3\alpha)}, \quad (35)$$

$$\begin{aligned} U_4(x,t) = {} & (50e^x t^{4\alpha}((-25-8e^x+170e^{2x}+248e^{3x}-425e^{4x}+200e^{5x})\Gamma(1+\alpha)^2\Gamma(1+2\alpha) - \\ & 24e^x(-1-5e^x+11e^{2x})\Gamma(1+2\alpha)^2 - 120e^x(-1+e^x+2e^{2x})\Gamma(1+\alpha)\Gamma(1+3\alpha))/ \quad (36) \\ & ((1+e^x)^8\Gamma(1+\alpha)^2\Gamma(1+\alpha)\Gamma(1+4\alpha)) \end{aligned}$$

The series solution is given by

$$U(x,t)=\frac{1}{(1+e^{x})^{2}}+\frac{10e^{x}t^{\alpha}}{(1+e^{x})^{3}\Gamma(1+\alpha)}+\frac{50e^{x}(-1+2e^{x})t^{2\alpha}}{(1+e^{x})^{4}\Gamma[(+2\alpha)}+U_{3}+U_{4}+\ldots \quad (37)$$

The exact solution, for the special case $\alpha=1$, is given by

$$U(x,t)=\frac{1}{(1+e^{x-5t})^{2}} \quad \textbf{[55].} \quad (38)$$

x	U_{ADM}	U_{HPSTM}	The exact solution	Absolute error ADM	Absolute error HPSTM
0.01	0.250231638	0.248753418	0.248751565	$-1\cdot 480x10^{-3}$	$-1\cdot 453x10^{-6}$
0.02	0.247698311	0.246265957	0.246264132	$-1\cdot 434x10^{-3}$	$-1\cdot 825x10^{-6}$
0.03	0.245184736	0.243791181	0.243789383	$-1\cdot 395x10^{-3}$	$-1\cdot 798x10^{-6}$
0.04	0.242689044	0.241329210	0.241327439	$-1\cdot 361x10^{-3}$	$-1\cdot 771x10^{-6}$
0.051	0.240210009	0.238880160	0.238878417	$-1\cdot 331x10^{-3}$	$-1\cdot 743x10^{-6}$

Table 1: The results of fractional fisher equation of 5 th order obtained by HPSTM with comparison with ADM and the exact solution at t=0.001 and α=1.

Application: 2. Nonlinear dispersive KdV equation

The discovery of solitary waves inspired scientists to conduct a huge size of research work to study this concept. Two Dutchmen Korteweg and deVries derived a nonlinear partial differential equation, well known by the KdV equation, to model the height of the surface of shallow water in the presence of solitary waves [56]. The KdV equation also describes the propagation of plasma waves in a dispersive medium. The nonlinear dispersive equation formulated by Korteweg and de Vries (KdV) in its simplest form [57] is given by:

$$D_{t}^{\alpha}U(x,t)+6U(x,t)U_{x}(x,t)+U_{xxx}(x,t)=0, 0<\alpha<1, t>0. \quad (39)$$

Subject to the initial condition

$$U(x,0)=\frac{1}{2}\sec h^{2}(\frac{1}{2}x) \quad (40)$$

As in the previous examples, we can apply HPSTM to obtain the solution $U(x,t)$ for $0<\alpha<1, t>0$, and we get:

$$p^{0}: U_{0}(x,t)=U(x,0)=\frac{1}{2}\sec h^{2}(\frac{1}{2}x)\ . \quad (41)$$

$$p^{1}: U_{1}(x,t)=-S^{-1}[u^{\alpha}S[U_{0xxx}+6U_{0}U_{0x}]] \quad (42)$$

After simplification with using mathematica package we have:

$$U_{1}(x,t)=\frac{t^{\alpha}(5+Cosh[x])Sech[\frac{x}{2}]^{4}Tanh[\frac{x}{2}]}{4\Gamma(1+\alpha)}, \quad (43)$$

$$U_{2}(x,t)=\frac{t^{2\alpha}(-307+42Cosh[x]+Cosh[2x])Sech[\frac{x}{2}]^{6}Tanh[\frac{x}{2}]^{2}}{16\Gamma(1+2\alpha)}, \quad (44)$$

$$U_3(x,t)=\frac{1}{256\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}(t^{3\alpha}((-205925+189404Cosh[x]-21980Cosh[2x]+292Cosh[3x]+ \quad (45)$$

$$Cosh[4x]\Gamma(1+\alpha)^2+12(-256+87Cosh[x]+24Cosh[2x]+Cosh[3x])\Gamma(1+2\alpha))Sech[\frac{x}{2}]^{10}Tanh[\frac{x}{2}])$$

$$U_4(x,t)=(t^{4\alpha}Sech[\frac{x}{2}]((-472338622+629524700Cosh[x]-180486881Cosh[2x]+19915326Cosh[4x]+$$

$$644194Cosh[4x]+2278Cosh[2x]+Cosh[6x])\Gamma(1+\alpha)^2\Gamma(1+2\alpha)+96(-52493+$$

$$61962Cosh[x]-10614Cosh[2x]-211Cosh[3x]+59Cosh[4x]+Cosh[5x])\Gamma(1+2\alpha)^2+ \quad (46)$$

$$96(23645-13414Cos[x]-798Cosh[2x]+70Cosh[3x]+Cosh[4x])\Gamma(1+\alpha)\Gamma(1+3\alpha)Sinh[\frac{x}{2}]^2))/$$

$$(4096\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+4\alpha))]$$

The solution is given by

$$U(x,t)=\frac{1}{2}\sec h^2(\frac{1}{2}x)+\frac{t^\alpha(5+Cosh[x])Sech[\frac{x}{2}]^4Tanh[\frac{x}{2}]}{4\Gamma[1+\alpha]}+U_2+U_3+U_4+\ldots \quad (47)$$

The exact solution, for the special case $\alpha=1$, is given by

$$U(x,t)=\frac{1}{2}\sec h^2(\frac{1}{2}(x-t)) \ [57]. \quad (48)$$

X	$U_{HPSTM}(\alpha=1)$	U_{exact}
0	0.500000	0.499988
1	0.397822	0.395037
2	0.212913	0.211590
3	0.091471	0.091174
4	0.035717	0.035666
5	0.013435	0.013428
6	0.004983	0.004982
7	0.001839	0.001839
8	0.000677	0.000677
9	0.000249	0.000249
10	0.000092	0.000092

Table2: The results obtained by HPSTM by 5th order approximate solution in comparison with the exact solution at $\alpha=1$ and t=0.01:

8 Conclusion

The aim of this work was to make use of the properties of Sumudu transform combined with Homotopy perturbation method that is called Homotopy perturbation Sumudu transform method (HPSTM). HPSTM has been utilized to derive the approximate analytical solutions for nonlinear fractional Fisher, fractional kdv equations. The method gives more realistic series solutions that converge very rapidly in physical problems. To demonstrate the validity of the proposed method, numerical results have been obtained which shows that the HPSTM strength lays in its ease of use and the possibility of using it as a tool to acquire approximate solutions of nonlinear fractional differential equation with excellent accuracy. Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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