IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 2 Issue 11, November 2015.
www.ijiset.com
ISSN 2348 – 7968

Separation Axioms in Bitopological Spaces

R. Suganya and N. Rajesh

ABSTRACT.
In this paper we introduce and study (i, j)-b-δ-R0 and (i, j)-b-δ- R1 spaces.

1. INTRODUCTION AND PRELIMINARIES

The concept of bitopological spaces was first introduced by Kelly [3]. After the introduction of the definition of a bitopological space by Kelly, a large number of topologists have turned their attention to the generalization of different concepts of a single topological space in this space. In this paper, we introduce and study the concept of weakly δ- b-continuous functions in bitopological spaces. Throughout this paper, the triple (X, τ1, τ2) where X is a set and τ1 and τ2 are topologies on X , will always denote a bitopological space. For a subset A of a bitopological space (X, τ1, τ2), the closure of A and the interior of A with respect to τi are denoted by i Cl(A) and i Int(A), respectively, for i = 1, 2. The notion of R_0 topological spaces introduced by Shanin
[6] in 1943. Davis [2] introduced the notion of R1 topological spaces which are independent of both T_0 and T_1 but strictly weaker than T_2. Some basic properties of the class of R_1 in topological spaces were discussed by Murdeshwar and Naimpally [5]. Bitopological forms of these concepts have appeared in the definitions of pairwise R_0 and sevic [4]. In this paper we introduce pairwise R1 spaces given by Mrsevic [4]. In this paper we introduce and study (i, j)-b-δ-R0 and (i, j)-b-δ-R1 spaces.

2. PRELIMINARIES

Definition 1. A subset A of a bitopological space (X, τ1 , τ2) is said to be
(1) (i, j)-regular open [1] if A = i Int(j Cl(A)), where i ≠ j, i, j = 1, 2,
(2) (i, j)-δ-open [1] if it is the union of (i, j)-regular open sets.

2000 Mathematics Subject Classification. 54D10.
Key words and phrases. Bitopological spaces, (i, j)-b-δ-closed set, (i, j)-b-δ-open set, (i, j)-b-δ-closure, (i, j)-b-δ-kernal.

(3) (i, j)-δ-b-open [7] if A ⊂ j Cl(i Intδ (A)) ∪ i Int(j Clδ (A)), where i ≠ j, i, j = 1, 2.

The complement of an (i, j)-regular open (resp. (i, j)-δ-open) set is called an (i, j)-regular closed (resp. (i, j)-δ-closed).

Definition 2. The intersection (resp. union) of all (i, j)-δ-closed (resp. (i, j)-δ-open) sets of X containing (resp. contained in) A ⊂ X is called the (i, j)-δ-closure (resp. (i, j)-δ- interior) of A and is denoted by (i, j)- Cl_δ (A) (resp. (i, j)-Int_δ (A)).

Definition 3. [7] A subset A of a bitopological space (X, τ1 , τ2) is said to be (i, j)-δ-b-open if A ⊂ j Cl(i Int_δ (A)) ∪ i Int(j Cl_δ (A)), where i ≠ j, i, j = 1, 2. The complement of an (i, j)-δ-b-open set is called an (i, j)- δ-b-closed set.

Definition 4. [7] The intersection (resp. union) of all (i, j)-δ-b-closed (resp. (i, j)-δ-b-open) sets of X containing (resp. contained in) A ⊂ X is called the (i, j)-δ-b-closure (resp. (i, j)-δ-b-interior) of A and is denoted by (i, j)-b Cl_δ (A) (resp. (i, j)-b Int_δ (A)). The intersection of all (i, j)-b-δ-open sets of X containing A is called the (i, j)-b-δ-kernal of A and is denoted by (i, j)-b Ker_δ (A).

Definition 5. [8] A bitopological space (X, τ1 , τ2) is said to be

(1) (i, j)-b-δ-T_0 if for every pair of distinct points in X , there exists an (i, j)-b-δ-open set of X containing one of the points but not the other.

(2) (i, j)-b-δ-T_1 if for every pair of distinct points x, y of X , there exists a pair of (i, j)-b-δ-open sets one containing x but not y and the other containing y but not x.

(3) (i, j)-b-δ-T_2 if for every pair of distinct points x, y of X , there exists a pair of disjoint (i, j)-b-δ-open sets, one containing x and the other containing y.

3. PROPERTIES OF (i, j)-b-δ-R_0 AND (i, j)-b-δ-R_1 SPACES

In this section, we introduce and study (i, j)-b-δ-R_0 and (i, j)-b-δ- R_1 spaces. These axioms can be characterized in terms of the (i, j)-b- δ-closure and the (i, j)-b-δ-kernal of singletons.

Definition 6. A bitopological space (X, τ1, τ2) is said to be (i, j)-b-δ- R_0 if for every (i, j)-b-δ-open set of X contains the (i, j)-b-δ-closure of each of its singletons.

Definition 7. A bitopological space (X, τ1, τ2) is said to be (i, j)-b-δ- symmetric if for each x, y ∈ X , x ∈ (i, j)-b Cl_δ ({y}) implies y ∈ (i, j)- b Cl_δ ({x}).

Theorem 3.1. A bitopological space (X, τ1, τ2) is (i, j)-b-δ-R_0 if, and only if it is (i, j)-b-δ-symmetric.

ISSN 2348 – 7968

Proof. Assume that (X, τ1, τ2) is (i, j)-b-δ-R_0. Let x ∈(i, j)-b $Cl_δ$ ({y}) and U be any (i, j)-b-δ-open set such that y ∈U . Then by hypothesis, x ∈U . Therefore, every (i, j)-b-δ-open set which contains y contains x. Hence, y ∈(i, j)-b $Cl_δ$ ({x}). Conversely, let U be an(i,j)-b-δ-open set and x ∈ U . If y ∉U , then x ∉(i, j)-b $Cl_δ$ ({y}), and thus by assumption, y ∉ (i, j)-b $Cl_δ$ ({x}). Therefore, (i, j)-b $Cl_δ$ ({x}) ⊂U , and hence, (X, τ1 , τ2) is (i, j)-b-δ-R_0.

Theorem 3.2. A bitopological space (X, τ1, τ2) is (i, j)-b-δ-T_1 if, and only if (X, τ1, τ2) is (i, j)-b-δ-T_0 and (i, j)-b-δ-R_0

Proof. Let x, y ∈ X and x ≠ y. Since (X, τ1, τ2) is (i, j)-b-δ-T_0, we may assume without loss of generality that x ∈G ⊂X \{y} for some (i, j)-b-δ-open set G. Thus, x ∉ (i, j)-b $Cl_δ$ ({y}), and so by Theorem 3.1, y ∉ (i, j)-b $Cl_δ$ ({x}). Therefore, X \(i, j)-b $Cl_δ$ ({x}) is an (i, j)-b-δ- open set containing y but not x. Hence, (X, τ1, τ2) is (i, j)-b-δ-T_1 . The converse is clear.

Proposition 3.3. For a bitopological space (X, τ1, τ2), the following statements are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-R_0 space;
(2) If for any F ∈(i, j)-BδC(X), x ∉ F , then F ⊂U and x ∉ U for some U ∈(i, j)-BδO(X);
(3) If for any F ∈(i, j)-BδC(X)such that x ∉ F , then F ∩ (i, j)-b $Cl_δ$ ({x})= ∅;
(4) If for any two distinct points x and y of (X, τ1, τ2), then either(i, j)-b $Cl_δ$ ({x}) = (i, j)-b $Cl_δ$ ({y}) or (i, j)-b $Cl_δ$ ({x}) ∩ (i, j)-b $Cl_δ$ ({y}) = ∅.

Proof. (1)⇒(2): Let F ∈ (i, j)-BδC(X) and x ∉F . Then by (1) (i, j)-b $Cl_δ$ ({x}) ⊂X \F . Set U = X \(i, j)-b $Cl_δ$ ({x}), then U ∈(i, j)-BδO(X) with F ⊂U and x ∉ U .

(2)⇒(3): Let F ∈(i, j)-$B_δ$C(X) such that x ∉ F . Then by (2), there exists U ∈ (i, j)-BδO(X) such that F ⊂U and x ∉U . Since U ∈(i, j)-BδO(X), U ∩ (i, j)-b Clδ ({x}) = ∅ and F ∩ (i, j)-b $Cl_δ$ ({x}) = ∅.

(3)⇒(4): Suppose that (i, j)-b $Cl_δ$ ({x}) ≠ (i, j)-b $Cl_δ$ ({y}) for the dis-tinct points x, y ∈ X . Then there exists z ∈ (i, j)-b $Cl_δ$ ({x}) such that z ∉(i, j)-b $Cl_δ$ ({y}) (or z ∈(i, j)-b $Cl_δ$ ({y}) such that z ∉(i, j)-b $Cl_δ$ ({x})). Then there exists V ∈ (i, j)-BδC(X, z) such that y ∉V , hence x ∈V . Therefore, x ∉(i, j)-b $Cl_δ$ ({y}). By (3), we obtain(i, j)-b Clδ ({x}) ∩ (i, j)-b $Cl_δ$ ({y}) = ∅. The proof for the other case is similar.

ISSN 2348 – 7968

(4)⇒(1): Let V ∈ (i, j)-BδO(X, x). For each y ∉V , we have x ≠y and x ∉ (i, j)-b Cl_δ ({y}). This shows that (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}). Hence by (4), (i, j)-b Cl_δ ({x}) ∩ (i, j)-b Cl_δ ({y}) = ∅ for each y ∈ X \ V and hence (i, j)-b Cl_δ ({x}) ∩ (∪ y∈X \V(i, j)-b Cl_δ ({y}))= ∅. On the other hand, since V ∈ (i, j)-BδO(X) and y ∈ X \ V ,we have (i, j)-b Cl_δ ({y}) ⊂ X \ V and hence X \ V = ∪ y∈X \V(i, j)-b Cl_δ ({y}). Therefore, we obtain (X \V) ∩ (i, j)-b Clδ ({x}) = ∅ and hence (i, j)-b Clδ ({x}) ⊂ V . This shows that (X, τ1 , τ2) is (i, j)-b-δ- R_0. 0

Theorem 3.4. A bitopological space (X, τ1, τ2) is (i, j)-b-δ-R_0 if, and only if for any x, y ∈ X , (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}) implies (i, j)-b Cl_δ ({x}) ∩ (i, j)-b Cl_δ ({y}) = ∅.

Proof. Suppose that (X, τ1, τ2) is (i, j)-b-δ-R_0 and x, y ∈ X such that(i, j)-b Cl_δ ({x}) ≠(i, j)-b Cl_δ ({y}). Then there exists z ∈ (i, j)-b Cl_δ ({x})such that z ∉(i, j)-b Cl_δ ({y}) (or z ∈ (i, j)-b Cl_δ ({y}) such that z ∉(i, j)-b Cl_δ ({x})). Since z ∉(i, j)-b Cl_δ ({y}), there exists V ∈ (i, j)-BδO(X, z) such that y ∉V . But z ∈ (i, j)-b Cl_δ ({x}) so x ∈ V. Therefore, x ∉(i, j)-b Cl_δ ({y}). Hence x ∈ X \ (i, j)-b Cl_δ ({y}) ∈(i, j)-BδO(X). Since (X, τ1, τ2) is (i, j)-b-δ-R_0, (i, j)-b Cl_δ ({x}) ⊂ X\ (i, j)-b Cl_δ ({y}). Hence (i, j)-b Cl_δ ({x}) ∩ (i, j)-b Clδ ({y}) = ∅. The proof for otherwise is similar. Conversely, let V ∈ (i,j)-BδO(X,x).Wewill show that (i, j)-b Cl_δ ({x}) ⊂ V . Let y ∉V ,that is, y ∈ X \ V.Then x = y and x ∉ (i, j)-b Cl_δ ({y}). This shows that (i, j)-b Cl_δ ({x})≠ (i, j)-b Cl_δ ({y}). By assumption, (i, j)-b Cl_δ ({x}) ∩ (i, j)-b Cl_δ ({y})= ∅. Hence y ∉(i, j)-b Cl_δ ({x}) and therefore (i, j)-b Cl_δ ({x}) ⊂ V .Hence (X, τ1 , τ2) is (i, j)-b-δ-R_0.

Theorem 3.5. A bitopological space (X, τ1, τ2) is (i, j)-b-δ-R_0 if, and only if for any points x and y in X , (i, j)-b Ker_δ ({x}) = (i, j)-b Ker_δ ({y}) implies (i, j)-b Ker_δ ({x}) ∩ (i, j)-b Ker_δ ({y}) = ∅.

Proof. Suppose that (X, τ1, τ2) is an (i, j)-b-δ-R_0 space. Then for any points x and y in X , if (i, j)-b Ker_δ ({x}) ≠ (i, j)-b Ker_δ ({y}), then (i, j)- b Clδ ({x}) ≠ (i, j)-b Cl_δ ({y}). Assume that z ∈ (i, j)-b Ker_δ ({x}) ∩ (i, j)-b Ker_δ ({y}). By z ∈ (i, j)-b Ker_δ ({x}), it follows that x ∈ (i, j)- b Cl_δ ({z}). Thus by Theorem 3.4, (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({z}).Similarly, we have (i, j)-b Clδ ({y}) = (i, j)-b Clδ ({z}) = ((i, j)-b Clδ ({x}), a contradiction. Hence, (i, j)-b Ker_δ ({x}) ∩ (i, j)-b Ker_δ ({y}) = ∅. Con- versely, let (X, τ1, τ2) be a bitopological space such that for any points x and y of X , (i, j)-b Ker({x}) ≠(i, j)-b Ker({y}) implies (i, j)-b Ker_δ ({x})∩ (i, j)-b Ker_δ ({y}) = ∅. Assume that (i, j)-b Cl_δ ({x}) ≠(i, j)-b Cl_δ ({y}).

Then (i, j)-b Ker_δ ({x}) ≠ (i, j)-b Ker_δ ({y}), and therefore by assumption, (i, j)-b Ker_δ ({x})

$\cap$ (i, j)-b Ker_δ ({y}) = $\emptyset$. Now if z $\in$ (i, j)-b Clδ ({x}), then x $\in$ (i, j)-b Ker_δ ({z}), and therefore, (i, j)-b Ker_δ ({x})$\cap$(i, j)-b Ker_δ ({z}) $\neq \emptyset$. By hypothesis, (i, j)-b Ker({x}) = (i, j)-b Ker_δ ({z}).Thus z $\in$ (i, j)-b Cl_δ ({x})$\cap$(i, j)-b Cl_δ ({y}) implies that (i, j)-b Ker_δ ({x}) =(i, j)-b Kerδ ({z}) = (i, j)-b Ker_δ ({y}), a contradiction. Therefore (i,j)-b Cl_δ ({x})$\neq$(i, j)-b Cl_δ ({y}) implies that (i, j)-b Cl_δ ({x})$\cap$(i, j)-b Cl_δ ({y}) =$\emptyset$, and thus by Theorem 3.4, (X, τ1, τ2) is (i, j)-b-δ-R_0.

Theorem 3.6. For a bitopological space (X, τ1, τ2), the following state- ments are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-R_0.
(2) For any nonempty subsets A of X and G $\in$ (i, j)-BδO(X) such that A $\cap$ G$\neq\emptyset$, there exists F $\in$ (i, j)-BδC(X) such that A $\cap$F = $\emptyset$ and F $\subset$G.
(3) For any G $\in$ (i, j)-BδO(X), G = $\cup${F : F $\in$ (i, j)-BδC(X), F$\subset$G}.
(4) For any F $\in$ (i, j)-BδC(X), F = $\cap${G : G $\in$ (i, j)-BδO(X),F $\subset$G}.
(5) For any x $\in$X , (i, j)-b Cl_δ ({x}) $\subset$(i, j)-b Kerδ ({x}).

Proof. (1)$\Rightarrow$(2): Let A be a nonempty set of X and G $\in$ (i, j)-BδO(X) such that A $\cap$ G $\neq \emptyset$. Then there exists x $\in$A $\cap$ G. Since x $\in$G $\in$ (i, j)-BδO(X), (i, j)-b Cl_δ ({x}) $\subset$G. Set F = (i, j)-b Cl_δ ({x}). Then F $\in$ (i, j)-BδC(X), F $\subset$G and A $\cap$ F = $\emptyset$.
(2)$\Rightarrow$(3): Let G $\in$ (i, j)-BδO(X), then G $\supset$ $\cup${F : F $\in$ (i, j)-BδC(X), F $\subset$G}. Let x be any point of G. Then there exists F $\in$ (i, j)-BδC(X) such that x $\in$F and F $\subset$G. Therefore, x $\in$ F $\subset$ $\cup${F : F $\in$ (i, j)- BδC(X), F $\subset$G}, and hence G = $\cup${F : F $\in$ (i, j)-BδC(X), F $\subset$G}.
(3)$\Rightarrow$(4): This is obvious.
(4)$\Rightarrow$(5): Let x be any point of X and y $\notin$(i, j)-b Ker({x}). Then there exists V $\in$ (i,j) -BδO(X, x) any y $\notin$V; hence (i, j)-b Cl_δ ({y})$\cap$ V = $\emptyset$. By (4), $\cap${G : G $\in$ (i, j)-BδO(X), (i, j)-b Cl_δ ({y}) $\subset$G},and hence there exists G $\in$ (i, j)-BδO(X) such that x $\notin$ G and (i, j)-b Cl_δ ({y}) $\subset$ G. Therefore, (i, j)-b Cl_δ ({x}) $\cap$ G = $\emptyset$ and hence y $\notin$ (i, j)-b Cl_δ ((i, j)-b Cl_δ ({x})) = (i, j)-b Cl_δ ({x}). Consequently, we obtain (i, j)-b Cl_δ ({x}) $\subset$(i, j)-b Ker({x}).
(5)$\Rightarrow$(1): Let G $\in$ (i, j)-BO(X, x). If y $\in$ (i, j)-b Ker({x}), then x $\in$(i, j)-b Cl_δ ({y}) and so y $\in$G. This implies that (i, j)-b Kerδ ({x}) $\subset$G.Therefore, x $\in$ (i, j)-b Cl_δ ({x}) $\subset$ (i, j)-b Kerδ ({x}) $\subset$G. This shows that (X, τ1, τ2) is an (i, j)-b-δ-R_0 space.

Corollary 3.7. For a bitopological space (X, τ1, τ2) is (i, j)-b-δ-R_0 if, and only if (i, j)-b Cl_δ ({x}) =(i, j)-b Ker_δ ({x}) for each x $\in$X .

Proof. Suppose that (X, τ1, τ2) is an (i, j)-b-δ-R_0 space. By Theorem3.6, (i, j)-b Cl_δ ({x}) ⊂ (i, j)-b Kerδ ({x}) for each x ∈ X . Let y ∈(i, j)-b Ker_δ ({x}). Then we have x ∈ (i, j)-b Cl_δ ({y}) and by Theorem3.4 (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}). Therefore, y ∈ (i, j)-b Cl_δ ({x})and hence (i, j)-b Ker_δ ({x}) ⊂ (i, j)-b Cl_δ ({x}). This shows that (i, j)-b Cl_δ ({x}) =(i, j)-b Ker({x}). The converse follows from Theorem 3.6.

Theorem 3.8. For a bitopological space (X, τ1, τ2), the following state- ments are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-R_0.

(2) x ∈ (i, j)-b Cl_δ ({y}) if, and only if y ∈ (i, j)-b Cl_δ ({x}) for anypoints x and y in X .Proof. (1)⇒(2): Assume that (X, τ1, τ2) is (i, j)-b-δ-R0 and x ∈ (i, j)- b Cl_δ ({y}). Then (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}). Hence y ∈ (i, j)- b Clδ ({x}). The other part is similar.(2)⇒(1): Let x ∈ U ∈ (i, j)-BδO(X, x). If y ∉U , then x ∉(i, j)-b Cl_δ ({y}) and hence y ∉ (i, j)-b Cl_δ ({x}) (by (2)). Thus (i, j)-b Cl_δ ({x}) ⊂U . Hence (X, τ1, τ2) is (i, j)-b-δ-R_0.

Theorem 3.9. For a bitopological space (X, τ1, τ2), the following state- ments are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-R_0.
(2) If F is an (i, j)-b-δ-closed subset of X , then F = (i, j)-b Kerδ (F).
(3) If F is an (i, j)-b-δ-closed subset of X and x ∈ F , then (i, j)-b Kerδ ({x}) ⊂ F .
(4) If x ∈ X , then (i, j)-b Kerδ ({x}) ⊂ (i, j)-b Clδ ({x}).

Proof. (1)⇒(2): Let F be an (i, j)-b-δ-closed subset of X and x ∉F . Thus X \F ∈ (i, j)-BδO(X, x). Since (X, τ1, τ2) is (i, j)-b-δ-R_0,(i, j)-b Cl_δ ({x}) ⊂ X \ F . Thus (i, j)-b Cl_δ ({x}) ∩ F = ∅ and x ∉(i, j)-b Ker(F). Therefore, (i, j)-b Kerδ (F) = F .
(2)⇒(3): In general, A ⊂ B implies (i, j)-b Kerδ (A) ⊂ (i, j)-b Kerδ (B). Therefore, it follows from (2) that (i, j)-b Ker({x}) ⊂ (i, j)-b Ker(F) = F .
(3)⇒(4): Since x ∈ (i, j)-b Cl_δ ({x}) and (i, j)-b Cl_δ ({x}) is (i, j)-b-δ- closed, by (3) (i, j)-b Kerδ ({x}) ⊂ (i, j)-b Cl_δ ({x}).
(4)⇒(1): Let x ∈ (i, j)-b Cl_δ ({y}). Then y ∈ (i, j)-b Kerδ ({x}). Sincex ∈ (i, j)-b Cl_δ ({x}) and (i, j)-b Cl_δ ({x}) is (i, j)-b-δ-closed, by (4) we obtain y ∈ (i, j)-b Kerδ ({x}) ⊂ (i, j)-b Cl_δ ({x}). Therefore, x ∈ (i, j)-b Cl_δ ({y}) implies that y ∈ (i, j)-b Cl_δ ({x}). Hence by Theorem 3.8, (X,τ1,τ2)is(i,j)-b-δ-R_0

Definition 8. A net $\{x_\alpha\}_{\alpha\in\Lambda}$ in a bitopological space (X, τ1, τ2) is called (i, j)-b-δconvergent to a point x in X if for every U $\in$ (i, j)-BδO(X, x), there exists $\alpha 0 \in \Lambda$ such that $x_\alpha \in U$ for each $\alpha \geq \alpha_0$.

Lemma 3.10. Let (X, τ1, τ2) be a bitopological space and let x and y any two points in X such that every net in X (i, j)-b-δ-converging to y (i, j)-b-δ-converges to x. Then x $\in$ (i, j)-b Cl_δ ({y}).

Proof. Suppose that xn = y for each n $\in$ N. Then {xn }n$\in$N is a net in X that (i, j)-b-δ-convergence to y. Thus by assumption, (i, j)-b-δ- converges to x. Hence x $\in$ (i, j)-b Cl_δ ({y}).

Theorem 3.11. For a bitopological space (X, τ1, τ2), the following state- ments are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-R_0.
(2) If x, y $\in$ X , then y $\in$ (i, j)-b Cl_δ ({x}) if, and only if every netin X (i, j)-b-δ-converging to y also (i, j)-b-δ-converges to x.

Proof. (1)⇒(2): Let x, y $\in$ X such that y $\in$ (i, j)-b Cl_δ ({x}). Suppose that $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X such that $\{x_\alpha\}_{\alpha\in\Lambda}$ (i, j)-b-δ-converges to y. Since y $\in$ (i, j)-b Cl_δ ({x}), by Theorem 3.1, x $\in$ (i, j)-b Cl_δ ({y}). Conversely, let x, y $\in$ X such that every net in X (i, j)-b-δ-converging to y (i, j)-b-δ-converges to x. Then x $\in$ (i, j)-b Cl_δ ({y}). By Theorem3.8, y $\in$ (i, j)-b Cl_δ ({x}).
(2)⇒(1): Assume that x and y are any two points of X such that(i, j)-b Cl_δ ({x}) ∩ (i, j)-b Cl_δ ({y}) ≠ ∅. Let z $\in$ (i, j)-b Cl_δ ({x}) ∩ (i, j)-b Cl_δ ({y}). So there exists a net {xα}α$\in$Λ in (i, j)-b Clδ ({x}) such that{xα }α$\in$Λ (i, j)-b-δ-converges to z. Since z $\in$ (i, j)-b Cl_δ ({y}), $\{x_\alpha\}_{\alpha\in\Lambda}$also (i, j)-b-δ-converges to y. Hence by (2) z $\in$ (i, j)-b Cl_δ ({y}). There-fore (i, j)-b Cl_δ ({z}) ⊂ (i, j)-b Cl_δ ({y}) (⋆). Hence y $\in$ (i, j)-b Cl_δ ({z}) gives (i, j)-b Cl_δ ({y}) ⊂ (i, j)-b Cl_δ ({z}) (⋆⋆). Hence from (⋆) and (⋆⋆),(i, j)-b Cl_δ ({y}) = (i, j)-b Cl_δ ({z}). Similarly it can be shown that (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({z}) by taking the net in (i, j)-b Cl_δ ({y}). So(i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}). By Theorem 3.4 (X, τ1, τ2) is (i, j)-b-δ-R_0 .

Definition 9. A bitopological space (X, τ1, τ2) is said to be (i, j)-b-δ- R_1 if for each points x and y of X such that (i, j)-b Cl_δ ({x}) = (i, j)- b Clδ ({y}), there exist disjoint (i, j)-b-δ-open subsets of X , say, U and V such that (i, j)-b Cl_δ ({x}) ⊂ U and (i, j)-b Cl_δ ({y}) ⊂ V .

Proposition 3.12. If a bitopological space (X, τ1, τ2) is (i, j)-b-δ-R1, then it is (i, j)-b-δ-R0 .

Proof. Let U $\in$ (i, j)-BδO(X, x). If y ∉ U , then x ∉ (i, j)-b Cl_δ ({y}). So (i, j)-b Cl_δ ({x}) = (i,j)-bCl_δ({y}). Since (X,τ1,τ2) is (i,j)-b-δ-R_1,

ISSN 2348 – 7968

there exists an (i, j)-b-δ-open set V_y such that (i, j)-b Cl_δ ({y}) ⊂ V_y andx ∉ V_y , which implies that y ∉ (i, j)-b Cl_δ ({x}). Hence (i, j)-b Cl_δ ({x})⊂ U . Therefore, (X, τ1 , τ2) is (i, j)-b-δ-R_0 .

Theorem 3.13. For a bitopological space (X, τ1, τ2), the following prop- erties are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-T_2,
(2) (X, τ1, τ2) is (i, j)-b-δ-R_1 and (i, j)-b-δ-T_1,
(3) (X, τ1, τ2) is (i, j)-b-δ-R_1 and (i, j)-b-δ-T_0.

Proof. (1)⇒(2): Since (X, τ1, τ2) is (i, j)-b-δ-T_2, then it is (i, j)-b-δ-T_1. Now if x, y ∈ X such that (i, j)-b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}), then x≠ y and there exist disjoint (i, j)-b-δ-open sets U and V such that x ∈ U and y ∈ V . Hence by Theorem 3.2, (i, j)-b Cl_δ ({x}) = {x} ⊂ U and (i, j)-b Cl_δ ({y}) = {y} ⊂ V . Hence (X, τ1 , τ2) is (i, j)-b-δ-R_1 .
(2)⇒(3): Since (X, τ1, τ2) is (i, j)-b-δ-T_1, then it is (i, j)-b-δ-T_0.
(3)⇒(1): Since (X, τ1 , τ2) is (i, j)-b-δ-R_1 and (i, j)-b-δ-T_0, then by Proposition 3.12, (X, τ1, τ2) is (i, j)-b-δ-R_0 and (i, j)-b-δ-T_0. Hence by Theorem 3.2, (X, τ1, τ2) is (i, j)-b-δ-T_1 . Let x, y ∈ X such that x ≠ y. Then (i, j)-b Cl_δ ({x}) = {x} ≠ {y} = (i, j)-b Cl_δ ({y}). Since (X, τ1, τ2) is (i, j)-b-δ-R_1 , there exist disjoint (i, j)-b-δ-open sets U and V such that (i, j)-b Cl_δ ({x}) = {x} ⊂ U and (i, j)-b Cl_δ ({y}) = {y} ⊂ V . Hence (X, τ1, τ2) is (i, j)-b-δ-T_2. and thus by Theorem 3.1 (X, τ) is an (i, j)-b-δ-R_0 space.

Corollary 3.14. For an (i, j)-b-δ-R_1 space (X, τ1 , τ2), the following statements are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-T_2.
(2) (X, τ1, τ2) is (i, j)-b-δ-T_1.
(3) (X, τ1, τ2) is (i, j)-b-δ-T_0.

Theorem 3.15. For a bitopological space (X, τ1, τ2) is (i, j)-b-δ-R_1 if, and only if x ∈ X \(i, j)-b Cl_δ ({y}) implies that x and y have disjoint (i, j)-b-δ-neighbourhoods.

Proof. Let x ∈ X \(i, j)-b Cl_δ ({y}). Then (i, j)-b Cl_δ ({x}) ≠ (i, j)- b Cl_δ ({y}) and x and y have disjoint (i, j) -b-δ-neighbourhoods. Con- versely, first we show that (X, τ1 , τ2) is (i, j)-b-δ-R_0. Let U be an(i, j)-b-δ-open set and x ∈ U . Suppose that y ∉U . Then, (i, j)-b Cl_δ ({y}) ∩ U = ∅ and x ∉(i, j)-b Cl_δ ({y}). Then there exist dis-joint (i, j)-b-δ-open sets U_x and U_y such that x ∈ U_x and y ∈ Uyand Ux ∩ U_y = ∅. Hence, (i, j)-b Cl_δ ({x}) ⊂ (i, j)-b Cl_δ (U_x) and (i, j)-b Cl_δ (x) ∩ Uy ⊂ (i, j)-b Cl_δ ({U_x }) ∩ U_y = ∅. Therefore, y ∉ (i,j)-b Cl_δ ({x}). Consequently, (i, j)-b Cl_δ ({x}) ⊂ U and (X, τ1, τ2) is (i, j)-b-δ-R_0 . Next, we show that (X, τ1 , τ2) is (i, j)-b-δ-R_1.

ISSN 2348 – 7968

Suppose that (i, j)-b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}). Then, we can assume that there exists z ∈ (i, j)-b Cl_δ ({x}) such that z ∉ (i, j)-b Cl_δ ({y}). Then there exist disjoint (i, j)-b-δ-open sets V_z and Vy such that z ∈ V_z , y ∈ V_y . Since z ∈ (i, j)-b Cl_δ ({x}), x ∈ V_z . Since (X, τ1 , τ2) is (i, j)-b-δ-R_0, we obtain (i, j)-b Cl_δ ({x}) ⊂ V_z , (i, j)-b Cl_δ ({y}) ⊂ Vy and V_z ∩ Vy = ∅. This shows that (X, τ1, τ2) is (i, j)-b-δ-R_1. and thus by Theorem 3.1 (X, τ) is an (i, j)-b-δ-R_0 space.

Theorem 3.16. For a bitopological space (X, τ1, τ2), the following state- ments are equivalent:

(1) (X, τ1, τ2) is (i, j)-b-δ-R_1.
(2) For each x, y ∈ X one of the following holds:
(a) If U is (i, j)-b-δ-open, then x ∈ U if, and only if y ∈ U .
(b) there exist disjoint (i, j)-b-δ-open sets U and V such that
x ∈ U and y ∈ V .
(3) If x, y ∈ X such that (i, j)-b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}), then there exist (i, j)-b-δ-closed sets F_1 and F_2 such that x ∈ F_1, y
∉ F1, y ∈ F_2, x ∉ F_2, and X = F_1 ∪ F_2 .

Proof. (1)⇒(2): Let x, y ∈ X . Then (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}) or (i, j)-b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}). If (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}) and U is (i, j)-b-δ-open, then x ∈ U implies y ∈ (i, j)-b Cl_δ ({x}) ⊂ U and y ∈ U implies x ∈ (i, j)-b Cl_δ ({y}) ⊂ U . Thus consider the case that (i, j)-b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}). Then there exist disjoint (i, j)-b-δ-open sets U and V such that x ∈ (i, j)-b Cl_δ ({x}) ⊂ U and y ∈ (i, j)-b Cl_δ ({y}) ⊂ V . (2)⇒(3): Let x, y ∈ X such that (i, j)-b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}). Then x ∉ (i, j)-b Cl_δ ({y}) or y ∉ (i, j)-b Cl_δ ({x}), say x ∉ (i, j)-b Cl_δ ({y}). Then there exists an (i, j)-b-δ-open set A such that x ∈ A and y ∉ A. Then by (2) there exist disjoint (i, j)-b-δ-open sets U and V such that x ∈ U and y ∈ V . Then F_1 = X \ V and F_2 = X \ U are (i, j)-b-δ-closed sets such that x ∈ F1, y ∉ F1, y ∈ F_2, x ∉ F_2 and X = F_1 ∪ F_2 .
(3)⇒(1): We shall first show that (X, τ1, τ2) is (i, j)-b-δ-R_0 space. Let U be an (i, j)-b-δ-open set such that x ∈ U . We claim that (i, j)- b Cl_δ ({x}) ⊂ U . For suppose y ∈ (i, j)-b Cl_δ ({x}) ∩ (X \U). Then (i, j)- b Cl_δ ({x}) ≠ (i, j)-b Cl_δ ({y}) (for if (i, j)-b Cl_δ ({x}) = (i, j)-b Cl_δ ({y}), then y ∈ U) and hence by (3), there exist (i, j)-b-δ-closed sets F_1 and F_2 such that x ∈ F_1, y ∉ F_1, y ∈ F_2, x ∉ F_2 and X = F_1 ∪ F_2 . Then y ∈ $F_2 \backslash F_1$ = X \$F_1$ ∈ (i, j)-BδO(X) and x ∉ X \$F_1$, a contradicts the fact that y ∈ (i, j)-b Cl_δ ({x}). Hence (X, τ1, τ2) is (i, j)-b-δ-R_0 space. Let p, q ∈ X be such that (i, j)-b Cl_δ ({p}) ≠ (i, j)-b Cl_δ ({q}). Then by the given condition there exist (i, j)-b-δ-closed sets H_1 and H_2 such that p ∈ H_1, q ∉ H_1, q ∈ H_2, p ∉ H_2 and X = H_1 ∪ H_2. Thus p ∈ $H_1 \backslash H_2$ and q ∈ $H_2 \backslash H_1$, where $H_1 \backslash H_2$ and $H_2 \backslash H_1$ are disjoint (i, j)-b-δ-open sets. Hence (i, j)-b Cl_δ ({p})⊂$H_1 \backslash H_2$and(i,j)-bCl_δ({q})⊂$H_2 \backslash H_1$.

Hence (X, τ1, τ2) is (i, j)-b-δ-R_1 space. and thus by Theorem 3.1 (X, τ) is an (i, j)-b-δ-R_0 space.

In view of Theorems 3.13 and 3.16, it now follows that

Theorem 3.17. A bitopological space (X, τ1, τ2) is (i, j)-b-δ-T_2 if, and only if for each x, y $\in$ X such that x $\neq$ y, there exist (i, j)-b-δ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

References

[1] G. K. Banerjee, On pairwise almost strongly θ-continuous mappings, Bull. Calcutta Math. Soc., 79(1987), 314-320.

[2] A. S. Davis, Indexed systems of neighbourhoods for general topological spaces, Amer. Math. Monthly, 68(1961), 886-893.

[3] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc., 13(1963), 71-89. sevic, On pairwise R0 and pairwise R1 bitopological spaces, Bull. Math.

[4] M. Mrsevic De la Soc. Sci. Mathe. de. la. R. S. de Roumanie Tome 30(78), nr.2, (1986).

[5] M. G. Murdeshwar and S. A. Naimpally, R1 topological spaces, Canad. Math. Bull. 9(1966), 521-523.

[6] N. A. Shanin, On separability in topological spaces, Dokl. Akad. Sciencies. USSR, 38(1943), 166-169.

[7] R. Suganya and N. Rajesh, Properties of δ-b-open sets in bitopological spaces (submitted).

[8] R. Suganya and N. Rajesh, Bitopological separatin axioms (submitted).