IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 2 Issue 12, December 2015.
www.ijiset.com
ISSN 2348 – 7968

Micropolar Elasticity Containing Voids

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Abstract

The present article is a theoretical study of micropolar elastic materials containing voids. The constitutive equation and linear field equation are derived. The coupling effect of micropolar and voids parameters are not considered in the internal energy function.

Key words: *Voids, Micro-inertia, Entropy, Equilibrated Inertia, Intrinsic equilibrated body force, Equilibrated stress vector.*

1 Introduction

Achenbach [1] discussed classical theory of wave propagation in elastic solid and typical phenomena such as radiation, reflection, refraction, diffraction and propagation in wave guides. Eringen [2-3] introduced the theory of micropolar continua, which is non-classical theory for material having certain kind of micro-structure, he also introduced linear theory of micropolar continua. Chanrasekharaiah [4] found that the linear version of the theory of micropolar thermoelasticity which admits second sound effect. Goodman and Cowin [5] established a continuum theory for granular materials, whose matrix or skeletal is elastic and interstices are voids. Nunziato and Cowin [6] studied non linear theory of elastic materials with voids by employing some basic ideas of Goodman-Cowin [5] theory. Cowin and Nunziato [7] developed linear theory of elastic materials with voids, in which a distributed body is written as the product of two fields, the material density field and volume fractional field. Marin [8] discussed some basic theorems in elastostatic of micropolar materials with voids. Iesan [9] established some general theorem on thermoelasticity of materials containing voids. Aouadi [10] derived the equations of the linear theory of thermoelastic diffusion in porous medium based on volume fractional field.
The present theory concern with elastic materials consisting of a distribution of voids. Following Goodman and Cowin [5] theory the voids volume distribution is included among the kinematics variables. In the present case of micropolar materials containing voids the coupling effect of voids and micro-structure of the material is not considered in the internal energy of the system. Thus using Helmholtz free energy the present theory is proposed for the determination of constitutive equations and field equations of the considered medium.

2 Basic equations

Definition: A material is called porous material if the voids are filled by some fluid having viscous nature. And a material is called voids material if the voids are filled by air or vacuum having no viscous nature.
Let us consider the region B_0 and its boundary ∂B_0 in three dimensional vector spaces occupied by Micropolar Elastic materials with voids in a reference configuration. And let B and ∂B be the region and its boundary at the current configuration. Let ν denotes the density of the matrix material, and φ be the volume distribution function, then mass density ρ of the region has the form:

$$\rho = \varphi\nu, \tag{1}$$

where, $0 < \nu \leq 1, \quad \nu = \nu(X,t), \quad \varphi = \varphi(X,t), \quad \rho = \rho(X,t)$.
At a reference state, the density or the voids distribution can be different, thus the equation (1) will take the form:

$$\rho_0 = \nu_0\varphi_0, \tag{2}$$

where ρ_0, ν_0, φ_0 are the equivalent functions to ρ, ν, φ in the reference configuration.
The energy balance of micropolar elastic materials with voids has the form [8]

$$\int_B \rho\left(\dot{u}_l\ddot{u}_k + j_{lk}\dot{\phi}_l\ddot{\phi}_k + \chi\dot{\varphi}\ddot{\varphi} + E\right)dV = \int_B \rho\left(f_l\dot{u}_l + l_l\dot{\phi}_l + l\dot{\varphi}\right)dV + \int_{\partial B}\left(T_l\dot{u}_l + M_l\dot{\phi}_l + h_l\dot{\varphi}\right)\partial S, \tag{3}$$

where V and S represent volume and surface area of the region B. This equation is valid for every time and every part of B.
Following Marin [8], the local balance of energy is given by

$$\rho\dot{E} = t_{lk}\dot{E}_{lk} + m_{lk}\dot{F}_{lk} - g\dot{\varphi} + h_l\dot{\varphi}_{,l}, \tag{4}$$

where t_{lk} and m_{lk} are stress tensor and couple stress tensor; ϵ_{ijk} is an alternating symbol; E_{lk} and F_{lk} are deformation and wryness tensor respectively; g is intrinsic equilibrated force and is introduced from the balance of equilibrated stress as follow (Nunziato and Cowin [6])

$$h_{l,l} + g + \rho l = \rho\chi\ddot{\varphi}. \tag{5}$$

The above symbols were described as in the referential configuration, although ρ and φ are bulk mass density and change in volume fraction from reference configuration. The superposed dot implies partial derivatives with respect to time variable, and the subscript preceded by coma implies covariant derivatives

3 Constitutive Equation

Following Nunziato and Cowin [6], we shall assumed that the internal energy of the material is independent of $\dot{\varphi}$, then $\Sigma = \{E_{ij}, F_{ij}, \varphi, \varphi_{,i}\}$ will be independent sets of constitutive variables. Thus the constitutive equations will take the forms:

$$t_{ij} = t_{ij}(\Sigma); \quad m_{ij} = m_{ij}(\Sigma); \quad g = g(\Sigma); \quad h_i = h_i(\Sigma); \quad \eta = \eta(\Sigma), \tag{6}$$

where η is entropy of the system.
Let us introduce Helmholtz free energy function $\psi = \psi(\Sigma)$ defined as:

$$\psi = E - \eta T, \tag{7}$$

where T is absolute temperature, it is time independent and assumed to be constant throughout the material. From equation (4) and (7), we obtain:

$$\rho\dot{\psi} = t_{lk}\dot{E}_{lk} + m_{lk}\dot{F}_{lk} - g\dot{\varphi} + h_l\dot{\varphi}_{,l} - \rho\dot{\eta}T, \tag{8}$$

now differentiating $\psi = \psi(\Sigma)$ we get,

$$\rho\dot{\psi} = \rho\frac{\delta\psi}{\delta E_{lk}}\dot{E}_{lk} + \rho\frac{\delta\psi}{\delta F_{lk}}\dot{F}_{lk} + \rho\frac{\delta\psi}{\delta\varphi}\dot{\varphi} + \rho\frac{\delta\psi}{\delta\varphi_l}\dot{\varphi}_{,l} \tag{9}$$

From equations (8) and (9) we obtain:

$$\left(t_{lk} - \rho\frac{\delta\psi}{\delta E_{lk}}\right)\dot{E}_{lk} + \left(m_{lk} - \rho\frac{\delta\psi}{\delta F_{lk}}\right)\dot{F}_{lk} - \left(g + \rho\frac{\delta\psi}{\delta\varphi}\right)\dot{\varphi} + \left(h_l - \rho\frac{\delta\psi}{\delta\varphi_{,l}}\right)\dot{\varphi}_{,l} - \rho\dot{\eta}T \;=\; 0, \tag{10}$$

the above relation (10) can be satisfied for arbitrary $\dot{E}_{lk}$, $\dot{F}_{lk}$, $\dot{\varphi}$ and $\dot{\varphi}_{,l}$ if and only if the corresponding coefficients vanish, provided

$$t_{lk} = \rho\frac{\delta\psi}{\delta E_{lk}}; \qquad m_{lk} = \rho\frac{\delta\psi}{\delta F_{lk}}; \qquad g = -\rho\frac{\delta\psi}{\delta\varphi}; \qquad h_l = \rho\frac{\delta\psi}{\delta\varphi_{,l}}; \qquad \dot{\eta} \;=\; 0, \tag{11}$$

these are the constitutive equations of micropolar elastic materials with voids.
We suppose that g is given as derivative of Helmholtz free energy, although some of the possibly desirable features of visco-elasticity are lost. It is because the function g still involve in $\dot{\varphi}$ and this lead to the behaviour of visco-elasticity, for simplicity we shall neglect $\dot{\varphi}$ from the sets of constitutive variables.

4 Linear theory

Expanding $\psi = \psi(E_{lk}, F_{lk}, \varphi, \varphi_{,i})$ by Quadratic approximation method we obtain;

$$\rho\psi = \rho\psi_0 + A_{lk}E_{ij} + B_{lk}F_{ij} + A\varphi + A_l\varphi_{,l} + \frac{1}{2}A_{ijlk}E_{ij}E_{lk} + \frac{1}{2}B_{ijlk}F_{ij}F_{lk} + \frac{1}{2}\zeta\varphi^2 + \frac{1}{2}C_{lk}\varphi_{,l}\varphi_{,l} + G_{ijkl}E_{ij}F_{kl}$$

$$+ H_{lk}E_{lk}\varphi + H_{lki}E_{lk}\varphi_{,i} + J_{lk}F_{lk}\varphi + J_{lki}F_{lk}\varphi_{,k} + L_l\varphi\varphi_{,l}, \tag{12}$$

at a reference state, the material is free from stress and couple stress with zero entropy, we shall have the following restriction for the above coefficients:

$$A_{lk} = B_{lk} = A = A_l = J_{lki} = 0. \tag{13}$$

Since the materials contain voids, and the voids volume distribution is partially through out the medium, thus we shall assumed that the interaction between voids and micro-rotations of the medium is no more (i.e. $J_{ij} = 0$). If the material is homogeneous, we shall have:

$$A_{ijkl} = A_{lkij}; \qquad B_{ijkl} = B_{lkij}; \qquad C_{lk} = C_{kl}, \tag{14}$$

using the above relation, equation (17) will take the form:

$$\rho\psi = \frac{1}{2}A_{ijkl}E_{ij}E_{kl} + \frac{1}{2}B_{ijkl}F_{ij}F_{kl} + \frac{1}{2}\zeta\varphi^2 + \frac{1}{2}C_{ij}\varphi_{,l}\varphi_{,k} + G_{ijkl}E_{ij}F_{kl}$$
$$+H_{lk}E_{lk}\varphi + H_{lki}E_{lk}\varphi_{,i} + L_l\varphi\varphi_{,l}, \tag{15}$$

thus, the constitutive equations (9) will take the forms:

$$t_{lk} = A_{ijkl}E_{lk} + G_{ijkl}F_{lk} + H_{lk}\varphi + H_{lki}\varphi_{,i},$$
$$m_{lk} = B_{ijkl}F_{lk} + G_{ijkl}E_{lk},$$
$$g = -\zeta\varphi - H_{lk}E_{lk} - L_l\varphi_{,l},$$
$$h_l = C_{lk}\varphi_{,l} + H_{lki}E_{lk} + L_l\varphi. \tag{16}$$

Let us define the deformation tensor and wryness tensor as:

$$E_{lk} = u_{l,k} + \epsilon_{lki}\phi_i \;; \qquad F_{lk} = \phi_{l,k}, \tag{17}$$

and the linearised forms of local balance of motion is defined as:

$$t_{lk} + \rho f_l = \rho\ddot{u}_l \tag{18}$$

$$m_{lk} + \epsilon_{lrs}t_{rs} + \rho l_l = \rho j_{lk}\ddot{\phi}_l. \tag{19}$$

In the theory of Nunziato and Cowin [6] the intrinsic equilibrated inertia χ is depend on the coordinate axes and time, but we shall follow Goodman and Cowin [5], assuming it to be constant for simplicity. Keeping mind equation (17), and putting equation (16) in to the equation (5), (18) and (19), we obtain the following coupled system:

$$A_{ijkl}(u_{i,jk} + \epsilon_{lkj}\phi_{j,k} + G_{ijkl}\phi_{i,jk}) + H_{lk}\phi_{,k} + H_{lkj}\phi_{j,k} + \rho f_l = \rho\ddot{u}_l;$$
$$G_{lkji}(u_{i,jk} + \epsilon_{lkj}\phi_{j,k}) + B_{lkji}\phi_{j,ik} - \epsilon_{lrs}\{H_{rs}\varphi + H_{rsk}\varphi_{,j} + A_{rsji}(u_{i,j} - \epsilon_{ijk}\phi_j) + G_{rsji}\phi_{j,i}\}$$
$$+\rho l_l = \rho j_{lk}\ddot{\phi}_k;$$
$$H_{lkj}(u_{j,kl} - \epsilon_{kjh}\phi_{h,l}) + L_l\varphi_{,l} + C_{lk}\varphi_{,lk} - H_{lk}(u_{k,l} - \epsilon_{lkh}\phi_h) - \zeta\varphi - L_l\varphi_{,l}$$
$$+\rho l = \rho\chi\ddot{\varphi}, \tag{20}$$

these equations forms a complete set of linear coupled system of field equations in the unknown u_i, ϕ_i and φ_i.

5 Isotropic Cases

If the media is isotropic, the constitutive coefficient will take the forms [2, 4].

$$H_{lk} = s\delta_{lk}; \qquad C_{lk} = a\delta_{lk}; \qquad j_{lk} = j\delta_{lk};$$

$$B_{lkji} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{lj}\delta_{ki} + \gamma\delta_{li}\delta_{jk};$$

$$A_{lkji} = \lambda\delta_{lk}\delta_{ji} + (\mu+\kappa)\delta_{lk}\delta_{ji} + \mu\delta_{il}\delta_{kj} \tag{21}$$

$$G_{lkji} = H_{lkj} = L_l = 0 \tag{22}$$

where, λ, μ are elastic constants, κ, α, β, γ are micropolar parameters, s and a are voids parameters. Then, the constitutive equation becomes:

$$t_{lk} = \mu(u_{l,k} + u_{k,l}) + \lambda u_{j,j}\delta_{lk} + \kappa(u_{l,k} + \epsilon_{lkj}\phi_j) + s\varphi\delta_{lk};$$

$$m_{lk} = \alpha\phi_{j,j}\delta_{lk} + \beta\phi_{l,k} + \gamma\phi_{l,k};$$

$$g = -su_{j,j} - \zeta\varphi\ ; \qquad h_i = a\varphi_{,l} \tag{23}$$

The field equations (20) will take the form:

$$(\mu+\kappa)u_{l,jj} + (\lambda+\mu)u_{j,jl} - \kappa\epsilon_{lkj}\phi_{j,k} + s\varphi_{,l} + \rho f_l = \rho\ddot{u}_l;$$

$$\gamma\phi_{l,jj} + (\alpha+\beta)\phi_{j,jl} + \kappa\epsilon_{lkj}u_{k,j} - 2\kappa\phi_l + \rho l_l = \rho j\ddot{\phi}_l;$$

$$a\varphi_{,jj} - su_{j,j} - \zeta\varphi + \rho l = \rho\chi\ddot{\varphi}, \tag{24}$$

this is the field equation for Micropolar elastic material containing voids. j and χ are positive, they are micro-inertia and equilibrated inertia respectively.

6 Uniqueness of the solution

Let P and K be Biot's potential and Kinetic energy defined respectively as

$$P(\Sigma) = E - \theta_0\eta, \tag{25}$$

$$K = \dot{u}_l\dot{u}_k + j_{lk}\dot{\phi}_l\dot{\phi}_k + \chi\dot{\varphi}^2, \tag{26}$$

using (12) and (13) in equation (25), we get

$$\rho P = \rho\psi = \frac{1}{2}A_{ijkl}E_{ij}E_{kl} + \frac{1}{2}B_{ijkl}F_{ij}F_{kl} + \frac{1}{2}\zeta\varphi^2 + \frac{1}{2}C_{ij}\varphi_{,l}\varphi_{,k} + G_{ijkl}E_{ij}F_{kl} +$$
$$H_{lk}E_{lk}\varphi + H_{lki}E_{lk}\varphi_{,i} + L_l\varphi\varphi_{,l}, \tag{27}$$

from (16-19) and (25-26) using divergence theorem, we obtain

$$\frac{d}{dt}\int_B \rho(K+P)dV = \int_B \rho\left(f_l\dot{u}_l + j_{lk}\dot{\phi}_l + \chi\dot{\varphi}\right)dV + \int_{\partial B}\left(t_{kl}\dot{u}_l m_{lk}\dot{\phi}_l + h_k\dot{\varphi}\right)n_k\partial S, \tag{28}$$

this is also the energy balance of the system.
Let us assigned the function $u_l(x,t)$, $\phi_l(x,t)$ and $\varphi(x,t)$ in the region B with its boundary ∂B for $x \in B$, and $t > 0$ for the solution of the equation (20). Now consider the following conditions:
(i) Initial conditions:

$$u_l = \dot{u}_l - \phi = \dot{\phi}_l = \varphi = 0; \text{ for } x \text{ in } B, \text{ and } t = 0. \tag{29}$$

(ii) Boundary conditions:

$$u_l = \dot{u}_l^* \text{ on } B_1;\ \phi_l = \dot{\phi}_l^* \text{ on } B_2;\ \varphi_l = \varphi_L^* \text{ on } B_2 \text{ and}$$

$$t_{lk}n_k = t_l^* \text{ on } B_1^c;\ m_{lk}n_k = m_l^* \text{ on } B_2^c;\ h_l n_k = h^* \text{ on } B_3^c;\ \text{for } t > 0, \tag{30}$$

where B_1, B_2, B_3 and B_1^c, B_2^c, B_3^c are the arbitrary parts of compliment of B. u_l^*, ϕ_l^*, φ_l^*, t_l^*, m_l^* and h^* are the prescribed function in the domains of their definitions.
Now we shall establish the uniqueness of the solution of equation (20). Using the above initial and boundary conditions, let us assumed that there exist at least one solution. In the absence of body forces and body couple forces, we shall consider that the boundary condition defined above is homogeneous having non-negative micro-inertia (j_{lk}) and Biot's potential (P), then the solution for the problem described above is trivial. Initially when $t = 0$ the equation (28) will be :

$$\frac{d}{dt}\int_B \rho(K+P)dV = 0 \tag{31}$$

using equations (27) and (29). We see that $K = P = 0$ for $t = 0$, consequently we conclude that the existence of unique solution.

7 Energy equation

To understand the equation (3) of the energy balance, we may integrate it over the region B, using divergence theorem, we see that:

$$\frac{d}{dt}\int_B \rho_0 E dV = \int_B (t_{ij}\dot{E}_{ji} + m_{ij}\dot{F}_{ij})dV - \int_B (g\dot{\varphi} + h_{i,i}\dot{\varphi})dV, \tag{32}$$

where B is the fixed body.
Employing equation (5) with $l = 0$ the above equation can be written as:

$$\frac{d}{dt}\int_B (\rho_0 E + \frac{\rho_0\chi\dot{\varphi}^2}{2})dV - \int_B (t_{ij}\dot{E}_{ji} + m_{ij}\dot{F}_{ij})dV = 0, \tag{33}$$

this equation is a generalized energy balance.

8 Conclusion

The presence of Voids and Porous in a micropolar elastic medium has been discussed. The cross term between voids distribution and micro-structure is not assumed, whereas this cross term may not be negligible in case of Porous medium, see Marin (1996). The constitutive equations and field equations are derived for the micropolar elastic medium containing voids. The uniqueness of the solution and energy equation are also discussed.

Acknowledgement

This work has been fulfilled by the financial support of North East India Tribal Minority Educational and Skills Development Society (NEITMESD Society), Aizawl.

References

1. Achenbach JD (1976) *Wave propagation in elastic solids.* North-Holland, Amsterdam.
2. Eringen AC (1966) Linear theory of micropolar elasticity. *Journal of Mathematics and Mechanics* 15, 909-249.
3. Eringen AC (1999) *Microcontinuum field theories I: Foundation and solids.* Springer Verlag, New York.
4. Chandrasekhariah DS (1986) Heat-flux dependent micropolar thermoelasticity. *International Journal of Engineering Science* 24, 1389-1395.
5. Goodman MA and Cowin SC (197) A continuum theory of granular materials. *Archive Rational Mechanics Analysis* 44, 249-266.
6. Nunziato JW and Cowin SC (1979) A non linear theory of elastic material with voids. *Archive Rational Mechanics Analysis* 72, 175-201.
7. Cowin SC and Nunziato JW (1983) Linear elastic material with voids. *Journal of Elasticity* 13, 125-147.
8. M. Marin (1996) Some basic theorem in elastostatics of micropolar materials with voids. *Journal of Computational and Applied Mathematics*, 70, 115-126.
9. Iesan D (1986) A theory of thermoelastic materials with voids. *Acta Mechanica* 60, 67-89.
10. Aouadi M (2010) A theory of thermoelastic diffusion materials with voids. *Journal of Applied Mathematics and Physics* 61, 357-379.