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On the fundamental theorems of calculus: A teaching approach.

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Abstract

The main objective of this paper is to give an alternative approach to teach some fundamental theorems in an introductory calculus course. Initially we approach, with an activity, the extreme value theorem for continuous functions of one real variable and after we can easily derive the intermediate value theorem, the theorems of Bolzano, Rolle and the mean value theorem. After, we propose a way of teaching the fundamental theorem of integral calculus. We believe this approach is more effective in presenting this topic in an introductory calculus class.

Keywords: *Extreme value theorem; Intermediate value theorem; Bolzano's theorem; Role's theorem; Mean value theorem. Fundamental theorem of integral calculus*

1. Introduction

One of the most important properties of continuous functions in the study of differential calculus is the fact that every continuous function on a closed interval must have both a maximum and a minimum value in this interval. In section 2 we approach this theorem with an activity and after we can easily derive some other fundamentals theorems.

In section 3 we propose a way of teaching the fundamental theorem of integral calculus. Usually to prove this theorem we use the existence of a primitive of a continuous function f in the closed interval [a, b] in the form of a definite integral

$$F(x) = \int_a^x f(t)dt \quad (1)$$

This fact creates difficulties for students. Therefore, before the proof of the theorem we propose two activities. The goal of the first one is to familiarize students with the approximate calculation of integrals and the second one on understanding naturally the proof of the fundamental theorem.

2. Fundamental theorems of differential calculus

2.1 The extreme value theorem

For the following functions find algebraically and graphically the sets f (A) and f (B).

(a) $$f(x) = 2x - 3 \quad (2)$$

With A= [-3, 4] and B= [2, +∞)

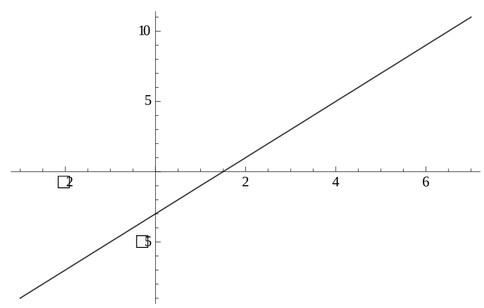

Fig.1: The function Eq. (1)

In this case: f (A) = [-9, 5] and f (B) = [1, +∞)

(b) $$f(x) = x^2 \quad (3)$$

With A= (-5, 5) and B= [-5, 5]

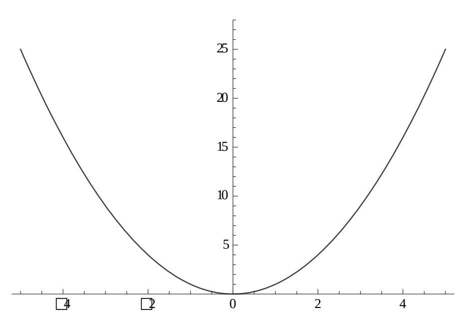

Fig.2: The function Eq. (2)

In this case: f (A) = [0, 25) and f (B) = [0, 25]

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(c) $$f(x)=\sqrt{x} \tag{4}$$

With A= [1, 4) and B= (1, 4]

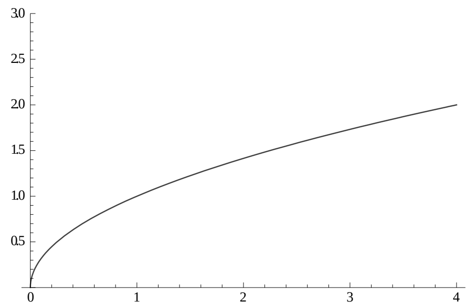

Fig.3: The function Eq. (3)

In this case: f (A) = [1, 2) and f (B) = (1, 2]

(d) $$f(x)=\begin{cases}\dfrac{1}{x}, x<0\\ \sqrt{x}, x\geq 0\end{cases} \tag{5}$$

With A= [-1, 4] and B= [1, 4]

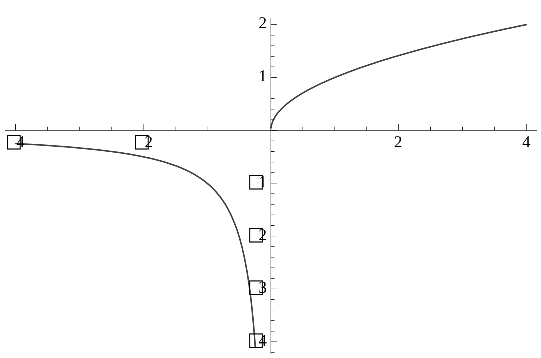

Fig.4: The function Eq. (4)

In this case: f (A) = [-1, - ∞) ∪ [0, 2] and f (B) = [1, 2]

In the previous activity we observe that:

(a) If f is a continuous function and if A is an interval, f (A) is also an interval.
(b) If A is a closed interval f (A) is also a closed interval.

We accept that this observation is true.

Theorem (Extreme value theorem): *If a real-valued function f is continuous in the interval A, then f (A) is also an interval. If A is a closed interval [a, b], then f(A) is a closed interval of the form [m, M].*

2.2 Some consequences

Consider an interval A= [a, b] in the real numbers ℝ and a continuous function f in this interval.

The intermediate value theorem: *If u is a number between f (a) and f (b), then there is c∈ (a, b) such that f(c) = u.*
Proof: f (a) and f (b) are elements of f (A) and by the previous theorem f (A) = [m, M]. By hypothesis u∈ [m, M], then there is c∈ (a, b) such that f(c) = u.

Theorem (Bolzano): *If f (a) f (b) < 0, then there is c ∈ (a, b) such that f(c) = 0.*
Proof: By hypothesis zero is between f (a) and f (b), then there is c∈ (a, b) such that f(c) = 0.

Theorem (Rolle): *If f is differentiable on the open interval (a, b) and f (a) = f (b), then there is c ∈ (a, b) such that* $f'(c)=0$

Proof: If f is a constant function the theorem is obvious. If the function f is not constant, it is sufficient to prove that there is an interior point of extremum (because in this case the theorem is a consequence of the Fermat's theorem). By hypothesis f is a continuous function, then f (A) = [m, M] and there is d, e ∈ [a, b] such that f (d) ≤ f(x) ≤ f (e). By hypothesis f (a) = f (b), then d or e is different from a or b and d or e is an interior point of extremum.

Mean value theorem: If f is differentiable on the open interval (a, b), then there exists some c in (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \tag{6}$$

Proof: Put g(x) = f(x) – d x, d∈ ℝ. g is continuous on [a, b] and differentiable on (a, b) with g′(x) = f′(x)-d. g (a) = g (b) if and only if $d=\dfrac{f(b)-f(a)}{b-a}$. With this value of d and the Rolle's theorem we obtain c in (a, b) such that Eq. (6) be true.

3. The fundamental theorem of integral calculus

3.1 Activity 1

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The goal of this activity is to familiarize students with the approximate calculation of integrals

- Calculate the left- hand sum s_{10} and the right-hand sum S_{10} for the integral

$$I = \int_0^1 t^2 dt \tag{7}$$

(a) With 10 subdivisions of the interval [0, 1].
(b) With n subdivisions of the interval [0, 1].
(c) Calculate the exact value of the integral I

Solution:
(a) Values of the function using 10 subdivisions of the interval [0, 1]:

Table 1: Values of the function $f(t) = t^2$

Value of t	***Value of t^2***
0	0
0.1	0.01
0.2	0.04
0.3	0.09
0.4	0.16
0.5	0.25
0.6	0.36
0.7	0.49
0.8	0.64
0.9	0.81
1	1

The left - hand sum:

$$\begin{aligned} s_{10} &= f(0)\Delta t + f(0.1)\Delta t + ... + f(0.9)\Delta t \\ &= (0 + 0.01 + 0.04 + ... + 0.81)0.1 \\ &= (2.85)0.1 = 0.285 \end{aligned} \tag{8}$$

The right- hand sum

$$\begin{aligned} S_{10} &= f(0.1)\Delta t + f(0.2)\Delta t + ... + f(1)\Delta t \\ &= (0.01 + 0.04 + ... + 0.81 + 1)0.1 \\ &= (3.93)0,1 = 0.393 \end{aligned} \tag{9}$$

Then

$$0.285 < \int_0^1 t^2 dt < 0.393 \tag{10}$$

(b) Using n subdivisions of the interval [0, 1] of length $\Delta t = 1/n$ with the numbers

$$t_0 = 0 < t_1 < t_2 < ... < t_n = 1,\ t_k = \frac{k}{n} \tag{11}$$

$$\begin{aligned} s_n &= f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n-1}{n}\right)\frac{1}{n} \\ &= \frac{1}{n}\left[0^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right] \\ &= \frac{1}{n^3}[1^2 + 2^2 + \cdots + (n-1)^2] \\ &= \frac{1}{n^3}\frac{(n-1)\cdot n(2n-1)}{6} = \frac{2n^2 - 3n + 1}{6n^2} \end{aligned} \tag{12}$$

(fig.1)

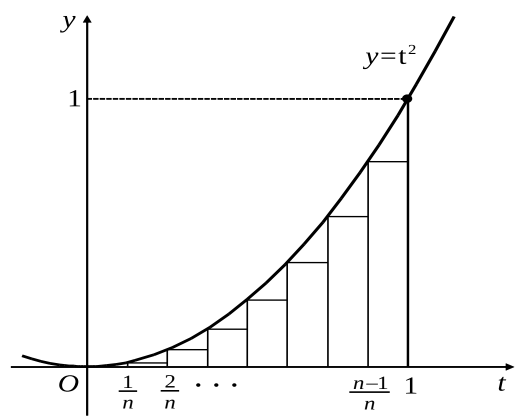

Figure 1: The left- hand sum s_n

And

$$\begin{aligned} S_n &= f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n}{n}\right)\cdot\frac{1}{n} \\ &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right] = \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2) \\ &= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{2n^2 + 3n + 1}{6n^2} \end{aligned} \tag{13}$$

(fig.2)

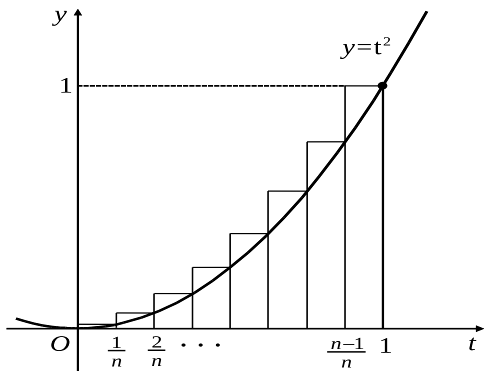

Figure 2: The right- hand sum S_n

(c) It is obvious that

$$s_n \le \int_0^1 t^2 dt \le S_n \qquad (14)$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = \frac{1}{3} \qquad (15)$$

Then

$$I = \int_0^1 t^2 dt = \frac{1}{3} \qquad (16)$$

3.2 Activity 2

The goal of this activity is the proof of the fundamental theorem of integral calculus:

Theorem: *If F is a primitive of a continuous function f in the interval [a, b]*

$$F(b) - F(a) = \int_a^b f(t)dt \qquad (17)$$

For this purpose we recommends to students to calculate the total change of a function in an interval, knowing the rate of change. In this way we come to the fact that the calculation of this change is equivalent to calculating the definite integral of the rate of change in this interval. We propose the following activity:

- We know that $F(t) = \frac{t^3}{3}$ is a primitive of $f(t) = t^2$ in the interval [0, 1].

(a) Make an underestimation of the total change $F(1) - F(0)$ using 10 subdivisions of the interval [0, 1]

(b) Compare the total change $F(1) - F(0)$ and

$$\int_0^1 t^2 dt \qquad (18)$$

(c) If F is a primitive of a continuous function f in the interval [a, b], using n subdivisions of the interval [a, b] of length Δt = (b-a)/n prove that Eq. (16) is true.

Solution:

(a)According to table 1

$$\begin{aligned} &F(1) - F(0) = F(1) - F(0.9) + F(0.9) - F(0.8) + ... \\ &... + F(0.1) - F(0) \\ &\approx F'(0.9)\times 0.1 + F'(0.8)\times 0.1 + ... + F'(0)\times 0.1 \\ &= f(0.9)\times 0.1 + f(0.8)\times 0.1 + ... + f(0)\times 0.1 \\ &= (0.81 + 0.64 + 0.49... + 0.01 + 0)\times 0.1 = 0.285 \end{aligned} \qquad (19)$$

Then

$$F(1) - F(0) \approx s_{10} \approx \int_0^1 f(t)dt \qquad (20)$$

(b) $F(1) - F(0) = \frac{1}{3}$, we observe that $F(1) - F(0) = \int_0^1 t^2 dt$

(c)Using n subdivisions of the interval [a, b] of length Δt = (b-a)/n

$$t_0 = a < t_1 < t_2 < ... < t_n = b \qquad (21)$$

We obtain

$$F(b) - F(a) = \sum_{k=1}^{n} [F(t_k) - F(t_{k-1})] \qquad (22)$$

And from the mean value theorem

$$F(t_k) - F(t_{k-1}) = f(\xi_k)\Delta t,\ \xi_k \in (t_{k-1}, t_k) \qquad (23)$$

Then

$$F(b) - F(a) = \sum_{k=1}^{n} f(\xi_k)\Delta t \qquad (24)$$

For n→+∞, we obtain the desired Eq. (16)

$$F(b) - F(a) = \int_a^b f(t)dt \qquad (25)$$

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4. Conclusions

In this paper we presented an alternative approach to teaching some basic theorems of calculus at an introductory level. Initially we approached, with an activity, the extreme value theorem for continuous functions of one real variable and after we easily proved the intermediate value theorem, the theorems of Bolzano, Rolle and the mean value theorem. After, we proposed a way of teaching the fundamental theorem of integral calculus. We believe this approach is more effective in presenting this topic in an introductory calculus class.

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