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A Study of Some Local Properties in Fuzzy Nearness Spaces

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ABSTRACT

In this paper we have studied some local properties of fuzzy nearness spaces. We have shown the equivalence of fuzzy completeness and fuzzy local completeness of a fuzzy nearness space. We also have shown that a fuzzy regular chained T1 FN-space is compact iff it is fuzzy uniformly locally compact and fuzzy uniformly chained.

1. Preliminaries : We present the basic definitions needed in the beginning. For detailed knowledge one should refer [1]. For any ordinary non-empty set X and I = [0,1], the collection I^X of all mappings from X to I is called the fuzzy space. $\mathcal{P}(I^X)$ denotes the powerset of I^X, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... denote members of I^X , μ , ξ , γ , ... denote the members of $\mathcal{P}(I^X)$.

A fuzzy nearness space (X, μ) consists of the fuzzy space I^X and a structure $\mu \subset \mathcal{P}(I^X)$ which satisfies the following axioms:

(FU1) ($\mathcal{A} \prec \mathcal{B}$ and $\mathcal{A} \in \mu$) $\Rightarrow \mathcal{B} \in \mu$,

(FU2) $\mathcal{A} \in \mu \Rightarrow$ for each $x \in X$, there is a pair $A_1, A_2 \in \mathcal{A}$ such that $A_1(x) + A_2(x) > 1$,

(FU3) $\phi \neq \mu \neq \mathcal{P}(I^x)$,

(FU4) $\mathcal{A}, \mathcal{B} \in \mu \Rightarrow \mathcal{A} \wedge \mathcal{B} \in \mu$,

(FU5) $\mathcal{A} \in \mu \Rightarrow \{int_\mu A \mid A \in \mathcal{A}\} \in \mu$, where

$$int_\mu A = \underline{1} - \inf \{\sigma \in Pt(I^x) \mid \{\sigma, A\} \in \mu\}.$$

The members of μ are called fuzzy uniform covers. The fuzzy map $f^{\rightarrow}: (X, \mu) \rightarrow (Y, \eta)$ is called fuzzy uniformly continuous if for each fuzzy uniform cover $\mathcal{A}$ of Y, $f^{\leftarrow}(\mathcal{A})$ is a fuzzy uniform cover of X where $f^{\leftarrow}(\mathcal{A}) = \{ f^{\leftarrow}(A) \mid A \in \mathcal{A}\}$.

A FN - space (X,ξ) is called a topological FN-space iff $\vee \{int_\mu A \mid A \in \mathcal{A}\} = \underline{1} \Leftrightarrow \mathcal{A} \in \mu$.

2. Definitions :

2.1 Definition : Let (X,μ) be a fuzzy nearness space. X is called **Fuzzy uniformly $\mathcal{U}$-Chained,** $\mathcal{U} \in \mu$, if there exists a positive integer n such that any two points of $Pt(I^X)$ can be joined by a $\mathcal{U}$-chain of length at most n. [i.e. for any $\rho, \sigma \in Pt(I^X)$, there is a finite sequence of points $\rho = \rho_1, \rho_2, ..., \rho_n = \sigma$ such that for each i=1,2,...,n there is $U \in \mathcal{U}$ with $\rho_i \vee \rho_{i+1} \leq U$] .

2.2 Definition : A FN-space (X,μ) is called **Fuzzy uniformly chained** if it is fuzzy uniformly $\mathcal{U}$ -chained for all $\mathcal{U} \in \mu$.

2.3 Definition :- Let $A \in I^X$ and $\mathcal{U} \subset I^X$ **.** Define **star of A in $\mathcal{U}$** denoted by **st(A, $\mathcal{U}$)** as $st(A,\mathcal{U}) = \vee\{U \in \mathcal{U} \mid U \wedge A \neq \underline{0}\}$.

2.4 Notation : $st^K(A,\mathcal{U}) = st(st^{K-1}(A, \mathcal{U}), \mathcal{U})$ where $st^1(A, \mathcal{U}) = st(A,\mathcal{U})$.

3. Some Local Properties : -

3.1 Lemma :- Let (X, μ) be a FN space. Let $\mathcal{U} \in \mu$. Then X is fuzzy uniformly $\mathcal{U}$-chained iff there exists $n \in N$ such that $st^n(\rho, \mathcal{U}) = \underline{1}$ for all $\rho \in Pt(I^X)$.

Proof :

Let a FN- Space (X, μ) be fuzzy uniformly $\mathcal{U}$ -chained.

Consider $st^n(\rho, \mathcal{U}) = \underline{1}$ for some $\rho \in Pt(I^X)$.

Let $A \leq \underline{1} - st^n(\rho, \mathcal{U})$.

Then there is $\sigma \in Pt(I^X)$ such that $\sigma \leq A$.

By F-$\mathcal{U}$-chainedness, there exist a sequence $\rho = \rho_1, \rho_2, ..., \rho_k = \sigma$ of fuzzy points in I^x and $U_1, U_2, ..., U_k$ in $\mathcal{U}$ such that $\rho_i \vee \rho_{i+1} \leq U_i$ for $1 \leq i \leq n-1$.

Therefore, $\sigma \leq U_k$ implies $A \wedge U_k \neq \underline{0}$. Thus $A \in st^{n+1}(\rho, \mathcal{U})$. By continuing for finitely many steps, we get $st^k(\rho, \mathcal{U}) = \underline{1}$.

Conversely, if there is $n \in N$ such that $st^n(\rho, \mathcal{U}) = \underline{1}$ for all $\rho \in Pt(I^X)$. Then any two distinct fuzzy points in I^X can be joined by a uniform cover $\mathcal{U} \in \mu$. Thus (X, μ) is fuzzy uniformly $\mathcal{U}$-chained.

3.2 Lemma : The fuzzy uniformly continuous image of a fuzzy uniformly chained nearness space is fuzzy uniformly chained.

Proof - Let (X, μ) and (Z, η) be two FN- spaces and let $f^{\rightarrow}: (I^X, \mu) \rightarrow (I^Y, \eta)$ be a fuzzy uniformly continuous function. Denote the image $f^{\rightarrow}(I^X) = I^Y$. Then (Y, η_Y) is a FN- space.

Let (X, μ) be fuzzy uniformly chained. Let $x_1, x_2 \in Pt(I^X)$ be any two fuzzy points.As $f^{\rightarrow}$ preserves fuzzy points $f^{\rightarrow}(x_1) = y_1$, $f^{\rightarrow}(x_2) = y_2$ are fuzzy points in $Pt(I^Y)$. For any $\mathcal{V} \in \eta_Y$, $f^{\leftarrow}(\mathcal{V}) \in \mu$. By fuzzy uniform chainedness of (X, μ) there exists $f^{\leftarrow}(V_1)$, $f^{\leftarrow}(V_2)$, ... , $f^{\leftarrow}(V_n)$ in $f^{\leftarrow}(\mathcal{V})$ such that $x_1 = x_{11}$, $x_{12}, \ldots, x_{1n} = x_2$ with $x_{1i} \vee x_{1i+1} \leq f^{\leftarrow}(V_i)$. Hence for i = 1, 2,..,n-1, $y_i \vee y_{i+1} \leq V_i$. Thus (Y, η_Y) is fuzzy uniformly $\mathcal{U}$-chained.

3.3 Lemma : The product of a family of fuzzy nearness spaces is fuzzy uniformly chained if and only if each factor space is.

<u>**Proof**</u> **:** Let $\{(I^{X_t}, \mu_t) \mid t \in T\}$ be a family of fuzzy uniformly chained FN-spaces. Denote $X = \Pi_{t \in T} X_t$. Consider the ordinary projection map $P_t: X \rightarrow X_t$ where $t \in T$. The fuzzy projection map from the fuzzy space I^X to I^{X_t} is the map $P_t^{\rightarrow}: I^X \varpi I^{X_t}$. The covering structure on the product space I^X is generated by

$\{P_t^{\leftarrow}(\mathcal{U}_t) \mid \mathcal{U}_t \in \mu_t,\ t \in T\}$.We know that the projection maps are uniformly continuous and onto. If (I^X, μ) is fuzzy uniformly chained then by lemma 3.2, for each $t \in T$, the image under $P_t^{\rightarrow}$ is uniformly chained. Thus each factor space is uniformly chained.For converse also the uniform continuity of $P_t^{\rightarrow}$ yields the result.

3.4 Lemma : If (X, μ) is a totally bounded and chained FN- space then it is fuzzy uniformly chained.

Proof : Let (X, μ) be a chained fuzzy nearness space which is totally bounded. By well chainedness of the FN-space (X, μ), for any two distinct points $\rho, \sigma \in Pt(I^X)$ there exists $\mathcal{U} \in \mu$ with a sequence $U_1, U_2, \ldots U_n$ in $\mathcal{U}$ and a sequence of fuzzy points $\rho = \rho_1, \rho_2, \ldots, \rho_k = \sigma$ in $Pt(I^X)$ such that for each $i = 1, 2, \ldots, n - 1$, $\rho_i \vee \rho_{i+1} \leq U_i$. By total boundedness of (X, μ), there is a finite $\mathcal{V} = \{V_1, V_2, \ldots, V_k\} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$. Thus $\rho_i \vee \rho_{i+1} \leq V_i$ for each $1 \leq i \leq n$. Hence (X, μ) is fuzzy uniformly chained.

3.5 Definition : A fuzzy nearness space (X, μ) is called **Fuzzy locally complete** if there is a fuzzy uniform cover of complete fuzzy subspaces of (X, μ).

3.6 Definition : A fuzzy nearness space (X, μ) is called **Fuzzy uniformly locally complete** if each fuzzy uniform cover of (X, μ) is refined by a fuzzy uniform cover of complete fuzzy subspaces of (X, μ).

3.7 Remark : The concept of fuzzy local completeness is weaker than that of fuzzy uniform local completeness.

3.8 Lemma : Every fuzzy closed subspace of a regular complete FN- space (X, μ) is complete

Proof : Let $A \subset X$. Let (X, μ) be regular complete FN-space. Let (A, μ_A) be a fuzzy closed subspace of (X, μ). Let $\mathcal{F}$ be a fuzzy Cauchy filter on I^A. Then $\mathcal{F}$ generates a fuzzy cauchy filter $\mathcal{G}$ on I^X .We know that a regular FN- space is complete iff each Cauchy fuzzy filter converges. Thus $\mathcal{G}$ Converges and has an adherence point. Consequently, $\mathcal{F}$ has an adherence point ρ.
I^A being closed, $\rho \in I^A$. Thus (A, μ_A) is fuzzy complete subspace of (X, μ).

3.9 Lemma : If a FN- space (X, μ) is fuzzy uniformly locally complete then it is fuzzy locally complete. The converse holds provided (X, μ) is regular.

<u>Proof</u> : (X, μ) being fuzzy uniformly locally complete, each $\mathcal{U} \in \mu$ is refined by $\mathcal{V} \in \mu$ where $\mathcal{V}$ contains fuzzy complete subspaces. Hence by definition, (X, μ) is fuzzy locally complete. For the converse, suppose (X, μ) is regular, fuzzy locally complete FN- space.

Let $\mathcal{U} \in \mu$ be a fuzzy uniform cover of fuzzy complete subspaces. By regularity of (X, μ) there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$ i.e. $\mathcal{V} = \{V \mid V <_\xi U \text{ for some } U \in \mathcal{U}\}$.

Then $\text{cl}\,\mathcal{V} = \{\text{cl } V \mid V \in \mathcal{V}\}$ and $\text{cl } V \leq U$ for each $V \in \mathcal{V}$ and some $U \in \mathcal{U}$.

i.e cl $\mathcal{V}$ refines $\mathcal{U}$.

Since cl V is a fuzzy closed subspace of a regular complete FN-space (X, μ) , clV is complete. Hence by definition, (X, μ) is fuzzy uniformly locally complete.

3.10 Remark : The following theorem indicates the equivalence of fuzzy completeness and fuzzy local completeness of a FN-space.

3.11 Theorem : A FN-space (X, μ) is fuzzy complete iff it is fuzzy locally complete.

Proof : Let (X, μ) be a complete FN-space. Then {1̲ } ∈ μ .Thus (X, μ) is fuzzy locally complete. Conversely, suppose (X, μ) is fuzzy locally complete. Then there is $\mathcal{U}$ ∈ μ such that each U ∈ $\mathcal{U}$ is a fuzzy complete subspace. Let $\mathcal{C}$ be a F- ξ -cluster on X. Then there is a $U_0 \in \mathcal{U}$ such that {U_0 , C} is q.c. for all C ∈ $\mathcal{C}$. But $\mathcal{C}$ is a F-grill. Hence $U_0 \in \mathcal{C}$.Now consider $\mathcal{F} = \{U_0 \wedge C \mid C \in \mathcal{C}\}$.
Then $\mathcal{F}$ ∈ ξ. Hence $\mathcal{F}$ is F-near on U_0. By the maximality of $\mathcal{C}$, $\mathcal{F}$ is maximal on U_0 . But U_0 is a complete fuzzy subspace. So the F-ξ-cluster on U_0 has an adherence point. Thus $\mathcal{C}$ has an adherence point. So (X, μ) is fuzzy complete.

3.12 Definition : A FN-space (X, μ) is called **Fuzzy locally compact** if it has a fuzzy uniform cover of fuzzy compact subspaces (i.e. fuzzy subspaces which are contigual and topological).

3.13 Definition : A FN-space (X, μ) is called **Fuzzy uniformly locally compact** if every fuzzy uniform cover is refined by a fuzzy uniform cover of fuzzy compact subspaces.

3.14 Theorem : Every locally compact FN-space is fuzzy complete.

Proof : Let (X, μ) be a locally compact FN-space. Then by definition, it has a fuzzy uniform cover of fuzzy compact subspaces. Let $\mathcal{U}$ ∈ μ where each U ∈ $\mathcal{U}$ is fuzzy compact. Then by definition, every family of F-closed fuzzy sets V ≤ U with finite q.c. property is q.c. and consequently has an adherence point. So U is complete fuzzy subspace. So $\mathcal{U}$ is a fuzzy uniform cover of complete fuzzy subspaces. (X, μ) is fuzzy locally complete. By theorem 3.11, (X, μ) is fuzzy complete.

3.15 Theorem : A Fuzzy dense subspace of a FN-space (X, μ) is fuzzy uniformly chained iff X is fuzzy uniformly chained.

Proof : Let $A \in I^X$. Suppuse A is fuzzy dense subspace of (X, μ) and is fuzzy uniformly chained. Then for each point $\rho \le A$, there is a positive integer n such that $A \le St^n(\rho, \mathcal{U})$ where $\mathcal{U} \in \mu$. If σ is not in A then $\sigma \le A^c$ since $A^C = \underline{1} - A$. Then there is $U_0 \in \mathcal{U}$ such that $\rho \le \text{int } U_0$. But A is dense implies $\{A, \text{int } U_0\}$ is q.c. Thus $A \wedge \text{int } U_0 \ne \underline{0}$. Hence $St^n(\rho, \mathcal{U}) \wedge \text{int } U_0 \ne \underline{0}$.
Thus $\sigma \le \text{int } U_0 \le St^{n+1}(\rho, \mathcal{U})$. Thus for each point σ in I^X, $\sigma \le St^k(\rho, \mathcal{U})$.
$\therefore St^k(\rho, \mathcal{U}) = \underline{1}$. So (X, μ) is fuzzy uniformly $\mathcal{U}$ -chained, hence (X, μ) is fuzzy uniformly chained.

3.16 Remark : Now we show that a fuzzy regular T_1 chained fuzzy nearness space is compact iff it is fuzzy uniformly locally compact and fuzzy uniformly chained.

3.17 Theorem : Let (X, μ) be a regular T_1 FN-space which is fuzzy chained. Then (X, μ) is fuzzy compact iff it is fuzzy uniformly locally compact and fuzzy uniformly chained.

Proof : Let (X, μ) be a regular T_1 fuzzy chained FN-space. Let (X, μ) be fuzzy compact FN-space. Then (X, μ) is topological and contigual FN-space. Therefore (X, μ) is fuzzy totally bounded. From Lemma 3.4, (X, μ) is fuzzy uniformly chained . Obviously, (X, μ) is fuzzy uniformly locally compact.
Now, suppose (X, μ) is fuzzy uniformly locally compact and fuzzy uniformly chained. Let $A \in I^X$ be a compact fuzzy subset and $\mathcal{U}$ be a fuzzy uniform cover

whose members are fuzzy compact. By the regularity of X, there is $\mathcal{V} \in \mu$ such that $\mathcal{V} = \{V \mid V <_\xi U$ for some $U \in \mathcal{U}\}$. Then there is a finite subset $\{V_1, V_2, \ldots, V_n\}$ of $\mathcal{V}$ such that $A \leq \vee\{V_i \mid 1 \leq i \leq n\}$. There exist $U_1, U_2, \ldots, U_n$ in $\mathcal{U}$ such that $V_i <_\xi U_i$ for $1 \leq i \leq n$. For each i, denote $C_i = \{U_i, V_i^c\}$ then $C_i \in \mu$.

Let $\mathcal{C} = C_1 \wedge C_2 \wedge \ldots \wedge C_n$

Then $\mathcal{C} \in \mu$.

$st(A, \mathcal{C}) \leq \vee\{st(V_i, \mathcal{C}) \mid 1 \leq i \leq n\} \leq \vee\{U_i \mid 1 \leq i \leq n\}$ which is fuzzy compact. Thus cl st (A, $\mathcal{C}$) is fuzzy compact. Thus st (A, $\mathcal{C}$) is fuzzy compact. As (X, μ) is fuzzy uniformly chained, by lemma 3.1, for any $\rho \in Pt(I^x)$ and for any non-negative integer k, $st^k(\rho, \mathcal{C}) = \underline{1}$. Thus (X, μ) is fuzzy compact.

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